INTENSE SHEET BEAM STABILITY PROPERTIES FOR UNIFORM PHASE-SPACE DENSITY*

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Abstract

A self-consistent one-dimensional waterbag equilibrium $f_b^0(x, p_x)$ for a sheet beam propagating through a smooth focusing field is shown to be exactly solvable for the beam density $n_b^0(x)$ and space-charge potential $\phi^0(x)$. A closed Schrodinger-like eigenvalue equation is derived for small-amplitude perturbations, and the WKB approximation is employed to determine the eigenfrequency spectrum as a function of the normalized beam intensity $s_b = \widehat{\omega}_{pb}^2/\gamma_b^2\omega_{\beta\perp}^2$, where $\widehat{\omega}_{pb}^2 = 4\pi\widehat{n}_b e_b^2/\gamma_b m_b$ is the relativistic plasma frequency-squared and $\widehat{n}_b = n_b(x=0)$ is the on-axis number density of beam particles.

SHEET BEAM EQUILIBRIUM WITH UNIFORM PHASE-SPACE DENSITY

We consider an intense sheet beam [1], made up of particles with charge e_b and rest mass m_b, which propagates in the z-direction with directed kinetic energy $(\gamma_b - 1)m_b c^2$ and average axial velocity $V_b = \beta_b c = const.$ Here, $\gamma_b = (1-\beta_b^2)^{-1/2}$ is the relativistic mass factor, c is the speed of light *in vacuo*, and the beam is assumed to be uniform in the y- and z- directions with $\partial/\partial y = 0 = \partial/\partial z$. The beam is centered in the x - direction at $x = 0$, and transverse confinement is provided by an applied focusing force, $F_x^{foc} = -\gamma_b m_b \omega_{\beta\perp}^2 x$, with $\omega_{\beta\perp}^2 = const$ in the smooth focusing approximation. The transverse dimension of the sheet beam is denoted by $2x_b$, and planar, perfectly conducting walls are located at $x = \pm x_w$. The particle motion in the beam frame is assumed to be nonrelativistic, and we introduce the effective potential $\psi(x,t)$ defined by

$$\psi(x,t) = \frac{1}{2}\gamma_b m_b \omega_{\beta\perp}^2 x^2 + \frac{1}{\gamma_b^2} e_b \phi(x,t). \tag{1}$$

The Vlasov-Maxwell equations describing the self-consistent nonlinear evolution of $f_b(x, p_x, t)$ and $\psi(x,t)$ can be expressed as [2]

$$\left(\frac{\partial}{\partial t} + v_x \frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial p_x}\right) f_b = 0, \tag{2}$$

and

$$\frac{\partial^2 \psi}{\partial x^2} = \gamma_b m_b \omega_{\beta\perp}^2 - \frac{4\pi e_b^2}{\gamma_b^2}\int_{-\infty}^{\infty} dp_x f_b. \tag{3}$$

As an equilibrium example $(\partial/\partial t = 0)$ that is analytically tractable, we consider the choice of distribution function

$$F_b(H_\perp) = \frac{\widehat{n}_b}{(8\gamma_b m_b \widehat{H}_\perp)^{1/2}}\Theta(H_\perp - \widehat{H}_\perp), \tag{4}$$

where $H_\perp = p_x^2/2\gamma_b m_b + \psi^0(x)$ is the transverse Hamiltonian, $\Theta(x)$ is the Heaviside step-function, and $\widehat{n}_b$, $\widehat{H}_\perp$ are positive constants. Evaluating the number density $n_b^0(x) = \int_{-\infty}^{\infty} dp_x F_b(H_\perp)$, we readily obtain

$$n_b^0(x) = \begin{cases} \widehat{n}_b\left[1 - \psi^0(x)/\widehat{H}_\perp\right]^{1/2}, & -x_b < x < x_b, \\ 0, & |x| > x_b. \end{cases} \tag{5}$$

Here, the location of the beam edge $(x = \pm x_b)$ is determined from

$$\psi^0(x = \pm x_b) = \widehat{H}_\perp, \tag{6}$$

where $\psi^0(x=0) = 0$ is assumed. It is useful to introduce the effective Debye length λ_D defined by

$$\lambda_D^2 = \frac{\gamma_b^3 \widehat{H}_\perp}{4\pi\widehat{n}_b e_b^2} = \frac{1}{2}\frac{\gamma_b^2 \widehat{v}_0^2}{\widehat{\omega}_{pb}^2}. \tag{7}$$

Here, $\widehat{v}_0 = (2\widehat{H}_\perp/\gamma_b m_b)^{1/2}$ is the maximum speed of a particle with energy $\widehat{H}_\perp$ as it passes through $x = 0$. Substituting Eq. (5) into Eq. (3) then gives

$$\frac{\partial^2}{\partial x^2}\left(\frac{\psi^0(x)}{\widehat{H}_\perp}\right) = \frac{1}{\lambda_D^2}\left(\frac{1}{s_b} - \left[1 - \frac{\psi^0(x)}{\widehat{H}_\perp}\right]^{1/2}\right) \tag{8}$$

in the beam interior $(-x_b < x < x_b)$. Equation (8) is to be integrated subject to the boundary conditions $\left[\psi^0\right]_{x=0} = 0 = \left[\partial\psi^0/\partial x\right]_{x=0}$. For physically acceptable solutions to Eq. (8), the condition $\left[\partial^2\psi^0/\partial x^2\right]_{x=0} > 0$ imposes the requirement that s_b lies in the interval $0 < s_b < 1$, where $s_b = \widehat{\omega}_{pb}^2/\gamma_b^2\omega_{\beta\perp}^2$. The regime $s_b \ll 1$ corresponds to a low-intensity, emittance-dominated beam, whereas the regime $s_b \rightarrow 1$ corresponds to a low-emittance, space-charge-dominated beam. In solving Eq. (8), it is convenient to introduce the dimensionless variables defined by

$$X = \frac{x}{\lambda_D}, \qquad \widehat{\psi}^0(X) = \frac{\psi^0(x)}{\widehat{H}_\perp}. \tag{9}$$

Substituting Eq. (9) into Eq. (8), integrating once, and enforcing $\left[\psi^0\right]_{x=0} = 0 = \left[\partial\psi^0/\partial x\right]_{x=0}$, gives

$$\frac{1}{2}\left(\frac{d\widehat{\psi}^0}{dX}\right)^2 = \frac{1}{s_b}\widehat{\psi}^0 + \frac{2}{3}\left[(1-\widehat{\psi}^0)^{3/2} - 1\right] \tag{10}$$

in the interval $-x_b/\lambda_D \le X \le x_b/\lambda_D$. Equation (10) can be integrated exactly to determine X as a function of $(1-\widehat{\psi}^0)^{1/2} = n_b^0(X)/\widehat{n}_b$ [see Eq. (5)]. We express $X =$

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$\int_0^{\widehat{\psi}^0} d\widehat{\psi}^0/(d\widehat{\psi}^0/dX)$, change variables to $z = (1-\widehat{\psi}^0)^{1/2}$, and make use of Eq. (10). This gives [1, 3]

$$X = 3^{1/2} \int_{(1-\widehat{\psi}^0)^{1/2}}^{1} \frac{zdz}{[(1-z)(a^+ - z)(z-a^-)]^{1/2}}, \tag{11}$$

where a^+ and a^- are defined by

$$a^{\pm} = \frac{1}{4s_b}\{3 - 2s_b \pm [3(3+4s_b-4s_b^2)]^{1/2}\}. \tag{12}$$

From Eqs. (6) and (11) we obtain a closed expression for x_b/λ_D in terms of the normalized beam intensity s_b for the choice of equilibrium distribution function in Eq. (4). The areal density of the beam particles, $N_b = \int_{-x_b}^{x_b} dx n_b^0(x)$, for the density profile in Eq. (5) can be expressed as

$$N_b = 2\widehat{n}_b \int_0^{x_b} dx[1 - \psi^0(x)/\widehat{H}_\perp]^{1/2}. \tag{13}$$

Some algebraical manipulation that make use of Eqs. (9), (10) and (13) gives

$$\frac{N_b}{2\widehat{n}_b x_b} = 3^{1/2}\frac{\lambda_D}{x_b}\int_0^1 \frac{z^2dz}{[(1-z)(a^+-z)(z-a^-)]^{1/2}}, \tag{14}$$

where x_b/λ_D is determined from Eq. (11). Note that $N_b/2\widehat{n}_b x_b$ depends only on the dimensionless intensity parameter s_b. Typical normalized density profiles

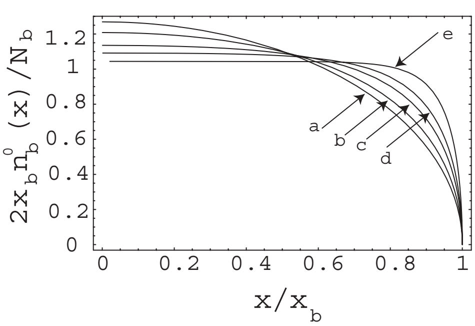

Figure 1: Plots of the normalized density profile $2x_b n_b^0(x)/N_b$ versus x/x_b for different values of the normalized beam intensity s_b corresponding to (a) $s_b = 0.2$, (b) $s_b = 0.9$, (c) $s_b = 0.99$, (d) $s_b = 0.999$, (e) $s_b = 0.999999$.

$2x_b n_b^0(x)/N_b$ are illustrated in Fig.1 for values of s_b ranging from $s_b = 0.2$ to $s_b = 0.999999$ [1]. Finally, defining the equilibrium transverse pressure profile by $P_b^0(x) = \int_{-\infty}^{\infty} dp_x (p_x^2/\gamma_b m_b) f_b^0$, we readily obtain

$$P_b^0(x) = \frac{4}{3}\widehat{n}_b \widehat{H}_\perp \left[1 - \frac{\psi^0(x)}{\widehat{H}_\perp}\right]^{3/2}. \tag{15}$$

Comparing Eqs. (5) and (15), note that $P_b^0(x) = const[n_b^0(x)]^3$, which corresponds to a triple-adiabatic pressure relation.

LINEARIZED EQUATIONS AND STABILITY ANALISIS

The *linearized* Vlasov-Maxwell equations can be expressed as [2]

$$\left(\frac{\partial}{\partial t} + v_x\frac{\partial}{\partial x} - \frac{\partial\psi^0}{\partial x}\frac{\partial}{\partial p_x}\right)\delta f_b = v_x \frac{\partial\delta\psi}{\partial x}\frac{\partial F_b}{\partial H_\perp}, \tag{16}$$

and

$$\frac{\partial^2}{\partial x^2}\delta\psi = -\frac{4\pi e_b^2}{\gamma_b^2}\delta n_b, \tag{17}$$

where $\delta n_b(x,t) = \int_{-\infty}^{\infty} dp_x \delta f_b$ is the perturbed number density of beam particles. In analyzing Eqs. (16) and (17), it is convenient to change variables from (x, p_x, t) to the new variables $(x', H_\perp, \tau)$ defined by [1]

$$x' = x, \qquad \tau = t, \qquad H_\perp = \frac{1}{2\gamma_b m_b}p_x^2 + \psi^0(x). \tag{18}$$

Substituting Eqs. (18) into Eqs. (16) and (17) gives for the evolution of the perturbations $\delta f_b(x', H_\perp, \tau)$ and $\delta\psi(x',\tau)$,

$$\left(\frac{\partial}{\partial \tau} + v_x\frac{\partial}{\partial x'}\right)\delta f_b = v_x \frac{\partial\delta\psi}{\partial x'}\frac{\partial F_b}{\partial H_\perp}, \tag{19}$$

$$\frac{\partial^2}{\partial x'^2}\delta\psi = -\frac{4\pi e_b^2}{\gamma_b^2}\delta n_b. \tag{20}$$

In Eq. (19), $v_x = +v(H_\perp, x')$ for the forward-moving particles with $v_x > 0$, and $v_x = -v(H_\perp, x')$ for the backward-moving particles with $v_x < 0$, where

$$v_x = \pm v(H_\perp, x') \equiv \pm\left(\frac{2H_\perp}{\gamma_b m_b}\right)^{1/2}\left[1 - \frac{\psi^0(x')}{H_\perp}\right]^{1/2}. \tag{21}$$

Furthermore,

$$\frac{\partial F_b}{\partial H_\perp} = -\frac{\widehat{n}_b}{2\gamma_b m_b \widehat{v}_0}\delta(H_\perp - \widehat{H}_\perp), \tag{22}$$

where $\widehat{v}_0 = (2\widehat{H}_\perp/\gamma_b m_b)^{1/2}$. Using Eqs. (19)-(22) and introducing $\delta E_x(x',\tau) = -(\partial/\partial x')\delta\phi(x',\tau) = -(\gamma_b^2/e_b)(\partial/\partial x')\delta\psi(x',\tau)$, after some algebraic manipulation we obtain [1]

$$\frac{\partial^2}{\partial\tau^2}\delta E_x - \widehat{v}_0^2 N(x')\frac{\partial}{\partial x'}\left[N(x')\frac{\partial}{\partial x'}\delta E_x\right] = -\frac{\widehat{\omega}_{pb}^2}{\gamma_b^2}N(x')\delta E_x, \tag{23}$$

where $N(x')$ is the (dimensionless) profile shape function defined by

$$N(x') = \left[1 - \frac{\psi^0(x')}{\widehat{H}_\perp}\right]^{1/2}. \tag{24}$$

In the analysis of Eq. (23), we make use of a normal-mode approach and express $\delta E_x(x',\tau) =$

$\delta\widehat{E}_x(x',\omega)\exp(-i\omega\tau)$, where ω is the (generally complex) oscillation frequency. Equation (23) can be represented in a convenient form by introducing the angle variable α defined by

$$\alpha = \frac{\pi}{2}\frac{X'}{X_b} = \frac{\omega_0}{\widehat{v}_0}X', \tag{25}$$

where X' and ω_0 are defined by

$$X' = \int_0^{x'} \frac{dx'}{N(x')}, \qquad \omega_0 = \frac{\pi}{2}\frac{\widehat{v}_0}{X_b}, \tag{26}$$

where $X_b = X'(x_b)$. Substituting Eq. (25) into Eq. (23) gives the eigenvalue equation

$$\omega_0^2 \frac{\partial^2}{\partial\alpha^2}\delta\widehat{E}_x + \left[\omega^2 - \frac{\widehat{\omega}_{pb}^2}{\gamma_b^2}N(\alpha)\right]\delta\widehat{E}_x = 0. \tag{27}$$

Equation (27) is to be solved over the interval $-\pi/2 < \alpha < \pi/2$ subject to the boundary conditions $\delta\widehat{E}_x(\alpha = \pm\pi/2,\omega) = 0$. Substituting Eqs. (10) and (24) into Eq. (25) gives

$$\alpha = \frac{\pi}{2}\frac{\lambda_D}{X_b}3^{1/2}\int_N^1 \frac{dz}{[(1-z)(a^+-z)(z-a^-)]^{1/2}}, \tag{28}$$

where $a^{\pm}$ is defined in Eq. (12). Some algebraical manipulation gives exactly for the inverse function $N(\alpha)$

$$N(\alpha) = \frac{\left[1 - a^+\kappa^2 sn^2\left(\frac{\alpha}{\pi}\frac{X_b}{\lambda_D}\left[\frac{a^+-a^-}{3}\right]^{1/2},\kappa\right)\right]}{\left[1 - \kappa^2 sn^2\left(\frac{\alpha}{\pi}\frac{X_b}{\lambda_D}\left[\frac{a^+-a^-}{3}\right]^{1/2},\kappa\right)\right]}, \tag{29}$$

where $sn(\beta,\kappa)$ is the Jacobi elliptic sine function and $\kappa = [(1-a^+)/(a^+-a^-)]^{1/2}$. In Eqs. (28)-(29), the "stretched" half-layer thickness (X_b) measured in units of the Debye length (λ_D) is given by

$$\frac{X_b}{\lambda_D} = \frac{2\cdot 3^{1/2}}{(a^+-a^-)^{1/2}}F\left(\arcsin(\kappa^2/a^+)^{-1/2},\kappa\right), \tag{30}$$

where F is the elliptic integral of the first kind. Using the expression for $N(\alpha)$ in Eq. (29), the eigenvalue equation (27) can be solved numerically for $\delta\widehat{E}_x(\alpha,\omega)$ and the eigenvalues ω^2 subject to the boundary conditions $\widehat{E}_x(\alpha = \pm\pi/2,\omega) = 0$. An approximate expression for the eigenvalues of the Schroedinger-like equation (27) can be obtained in the WKB approximation. The Born-Zommerfeld formula, when applied to Eq. (27), gives

$$\frac{\widehat{\omega}_{pb}}{\gamma_b\omega_0}\int_{-\pi/2}^{\pi/2} d\alpha\left[\left(\frac{\gamma_b\omega_m}{\widehat{\omega}_{pb}}\right)^2 - N(\alpha)\right]^{1/2} = \pi m, \tag{31}$$

where ω_m is the mth-mode eigenfrequency with m half-wavelength oscillations of $\delta\widehat{E}_x$ over the layer thickness. Making use of Eq. (28), the result in Eq. (31) can be rewritten as

$$6^{1/2}\int_0^1 \frac{dz(q_m^2-z)^{1/2}}{[(1-z)(a^+-z)(z-a^-)]^{1/2}} = \pi m, \tag{32}$$

where q_m and r are defined by $q_m = \omega_m/(\widehat{\omega}_{pb}/\gamma_b)$ and $r = \kappa[(q_m^2-a^+)/(q_m^2-1)]^{1/2}$. Equation (32) has been solved numerically [1] for ω_m^2, and the results have been compared with the numerical solutions of the eigenvalue equation (27) (Fig. 2). In Fig. 2, the convention is such that there are m half-wavelength oscillations of $\delta\widehat{E}_x$ over the layer thickness. Note that low beam intensity $(s_b \ll 1)$ corresponds to $\nu/\nu_0 \to 1$, with $\omega_m \simeq m\omega_{\beta\perp}$, whereas the space-charge-dominated regime $(s_b \to 1)$ corresponds to $\nu/\nu_0 \to 0$, with $\omega_m \simeq \omega_{\beta\perp} \simeq \widehat{\omega}_{pb}/\gamma_b$.

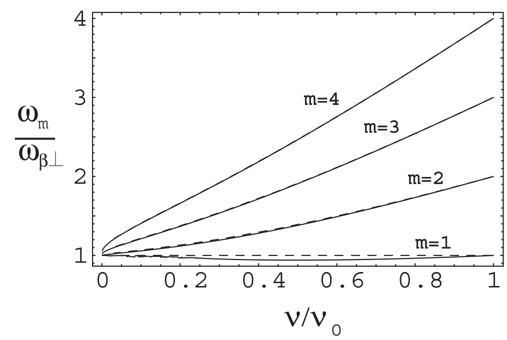

Figure 2: Plots of the normalized mode frequencies $\omega_m/\omega_{\beta\perp}$ versus the on-axis $(x = 0)$ tune depression $\nu/\nu_0 = (1-s_b)^{1/2}$ for several values of mode numbers $m = 1,2,3,4$. The dotted curves are the numerical solutions of the eigenvalue equation (27); the solid curves are the solutions obtained in the WKB approximation [Eq. (32)].

To summarize, we have demonstrated that the self-consistent waterbag equilibrium f_b^0 satisfying the steady-state $(\partial/\partial t = 0)$ Vlasov-Maxwell equations is exactly solvable for the beam density $n_b^0(x)$ and electrostatic potential $\phi^0(x)$. In addition, we derived a closed Schroedinger-like eigenvalue equation for small-amplitude perturbations $(\delta f_b, \delta\phi)$ about the self-consistent waterbag equilibrium in Eq. (4). In the eigenvalue equation, the density profile $n_b^0(x)$ plays the role of the potential $V(x)$ in the Schroedinger equation. The eigenvalue equation was investigated analytically and numerically, and the eigenfrequencies were shown to be purely real.

REFERENCES

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[2] R. C. Davidson and H. Qin, *Physics of Intense Charged Particle Beams in High Energy Accelerators* (World Scientific, Singapore, 2001), and references therein.

[3] The integrals in Eqs. (11), (14), (28), (32) can be expressed in terms of elliptic functions (see Ref. 1).