LINEAR COUPLING OF RMS EMITTANCES*

L.C. Teng[#], Argonne National Laboratory, Argonne, IL 60439 USA

Abstract

A general formulation of the linearly coupled rms emittances in two degrees of freedom is given. This formulation shows clearly what can be done to the emittances and how best to design for the necessary coupling.

NOTATION AND DEFINITIONS

The phase point of the ith particle in a distribution (beam) is represented by a column vector

$$X_i \equiv \begin{pmatrix} x_i \\ x_i' \end{pmatrix}. \tag{1}$$

(The index i is often omitted as being understood.) Its symplectic conjugate (a row vector) is defined as

$$X^+ \equiv \tilde{X}\tilde{S} = (x\ x')\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = (x' - x), \tag{2}$$

where as shown S is the unit symplectic matrix. The symplectic conjugate of a row vector such as X^+ is defined as

$$\left(X^+\right)^+ \equiv S\widetilde{X^+} = -X. \tag{3}$$

For a distribution of phase points (particles) the second-moment matrix is defined as the outer product

$$E \equiv -\overline{XX^+} = \begin{pmatrix} -\overline{xx'} & \overline{x^2} \\ -\overline{x'^2} & \overline{xx'} \end{pmatrix}, \tag{4}$$

where a bar means averaging over the distribution. We see immediately Tr(E) = 0 and E^+ = -E. We can parameterize E as

$$E = \varepsilon\begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \equiv \varepsilon J. \tag{5}$$

Matrix J is the Courant-Snyder "imaginary" unit matrix with |J| = 1 and J^2 = -I. The rms emittance ε is then given by

$$\varepsilon^2 \equiv |E| = \overline{x^2}\ \overline{x'^2} - \overline{xx'}^2. \tag{6}$$

We write the transfer matrix for the motion as

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{7}$$

and define its symplectic conjugate as

$$M^+ \equiv S\tilde{M}\tilde{S} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{8}$$

Then $\left(M^+\right)^+$ = M, and

$$MM^+ = M^+M = |M|I, \quad M+M^+ = [\mathrm{Tr}\ (M)]I. \tag{9}$$

If $M^+M = M^+M = I$, M is called symplectic and |M| = 1.

The transformations given by M are

$$X_T = MX, \quad X_T^+ = X^+M^+ \tag{10}$$

and

$$E_T = MEM^+, \quad \varepsilon_T^2 = |E_T| = |M|^2\varepsilon^2. \tag{11}$$

Thus, the emittance is invariant for a symplectic transformation.

The second-moment (rms) phase ellipse is defined as

$$X^+E^{-1}X = -1, \tag{12}$$

where X (without index i) is now the running variable. Since $E^{-1} = \frac{E^+}{|E|} = \frac{-E}{|E|}$ we can also write the ellipse as

$$X^+EX = |E| \tag{13}$$

or

$$X^+JX = \gamma x^2 + 2\alpha xx' + \beta x'^2 = \varepsilon. \tag{14}$$

We can diagonalize E^{-1} to get the area of the ellipse and show that

$$\varepsilon = |E|^{\frac{1}{2}} = \frac{1}{\pi}(\text{area of ellipse}). \tag{15}$$

TWO DEGREES OF FREEDOM

With two degrees of freedom (dof) we will write all 4-dimensional (4-D) vectors and matrices in the "block form." The phase-point position vector is now

$$\mathbf{X}_i \equiv \begin{pmatrix} x_i \\ x_i' \\ y_i \\ y_i' \end{pmatrix} \equiv \begin{pmatrix} X_i \\ Y_i \end{pmatrix}. \tag{16}$$

The symplectic conjugate is then

$$\mathbf{X}^+ = (\tilde{X}\ \ \tilde{Y})\begin{pmatrix} \tilde{S} & 0 \\ 0 & \tilde{S} \end{pmatrix} = \left(X^+\ \ Y^+\right). \tag{17}$$

As before,

$$\left(\mathbf{X}^+\right)^+ \equiv \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} \widetilde{X^+} \\ \widetilde{Y^+} \end{pmatrix} = -\mathbf{X}. \tag{18}$$

For a distribution, the second-moment matrix is

*Work supported by the U.S. Department of Energy, Office of Basic Energy Sciences, under Contract No. W-31-109-ENG-38.
[#]teng@aps.anl.gov

$$\mathbf{E} \equiv -\overline{\mathbf{XX}^{+}} = \begin{pmatrix} -\overline{XX^{+}} & -\overline{XY^{+}} \\ -\overline{YX^{+}} & -\overline{YY^{+}} \end{pmatrix} \equiv \begin{pmatrix} E & K \\ -K^{+} & F \end{pmatrix}, \quad (19)$$

where E and F retain their 1-dof forms and the coupling block is

$$K \equiv -\overline{XY^{+}} = \begin{pmatrix} -\overline{xy'} & \overline{xy} \\ -\overline{x'y'} & \overline{x'y} \end{pmatrix}, \qquad \mathrm{Tr}\,(K) \neq 0\,. \quad (20)$$

The projection of all the phase points on the, say, X-plane will have a distribution given simply by X_i, hence the "projection" rms emittance and second-moment ellipse are as given in Eqs. (6) and (13).

We can define the 4-D second-moment "ellipsoid" (bounded by a closed 3-D surface) as

$$\mathbf{X}^{+}\mathbf{E}^{-1}\mathbf{X} = -1\,. \quad (21)$$

In block form we have

$$\mathbf{E}^{-1} = \frac{1}{e}\begin{pmatrix} A & -C^{+} \\ C & B \end{pmatrix}, \quad (22)$$

where

$$\begin{cases} A = |F|E^{+} + KFK^{+} = -A^{+} \\ B = |E|F^{+} + K^{+}EK = -B^{+} \\ C = |K|K^{+} + FK^{+}E \end{cases} \quad (23)$$

and

$$e = \frac{|A|}{|F|} = \frac{|B|}{|E|} = \frac{|C|}{|K|} = |E||F| + |K|^{2} + \mathrm{Tr}\left(EKFK^{+}\right).$$

The equation of the "ellipsoid" in block form is then

$$X^{+}AX - X^{+}C^{+}Y + Y^{+}CX + Y^{+}BY = -e. \quad (24)$$

The Liouville invariant is defined as

$$\mathscr{L} \equiv |\mathbf{E}|^{\frac{1}{2}} = \frac{2}{\pi^{2}} \text{ (4-D volume of the ellipsoid)} \quad (25)$$

but has nothing to do with emittances. The "projection" ellipses are the projections of the ellipsoid on the X and Y planes, and their areas are related to the "projection" emittances as before.

PROPOGATION OF EMITTANCES AND INVARIANTS

The linearly coupled 2-dof motion is given by a transfer matrix

$$\mathbf{M} \equiv \begin{pmatrix} M & m \\ n & N \end{pmatrix}. \quad (26)$$

The symplectic conjugate of $\mathbf{M}$ is defined as

$$\mathbf{M}^{+} \equiv \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} \tilde{M} & \tilde{n} \\ \tilde{m} & \tilde{N} \end{pmatrix}\begin{pmatrix} \tilde{S} & 0 \\ 0 & \tilde{S} \end{pmatrix} = \begin{pmatrix} M^{+} & n^{+} \\ m^{+} & N^{+} \end{pmatrix}. \quad (27)$$

Thus,

$$\left(\mathbf{M}^{+}\right)^{+} = \mathbf{M}\,. \quad (28)$$

$\mathbf{M}$ is symplectic if

$$\mathbf{MM}^{+} = \mathbf{M}^{+}\mathbf{M} = I. \quad (29)$$

This gives in terms of the block matrices

$$\begin{cases} |M| = |N|, \quad |m| = |n|, \quad |M| + |m| = 1 \\ m^{+}M + N^{+}n = 0 \end{cases}. \quad (30)$$

The propagated (transformed) second-moment matrix is then

$$\mathbf{E}_T = \mathbf{MEM}^{+}\,. \quad (31)$$

This gives in block form

$$\begin{cases} E_T = MEM^{+} + mFm^{+} + MKm^{+} - mK^{+}M^{+} \\ F_T = NFN^{+} + nEn^{+} - NK^{+}n^{+} + nKN^{+} \\ K_T = MKN^{+} - mK^{+}n^{+} + MEn^{+} + mFN^{+} \end{cases}, \quad (32)$$

which then gives

$$\begin{cases} \varepsilon_{xT}^{2} = |E_T| = |M|^{2}\varepsilon_x^{2} + |m|^{2}\varepsilon_y^{2} + 2|M||m|\kappa \\ \qquad - 2|M|\mathrm{Tr}\left(EKm^{+}M\right) - 2|m|\mathrm{Tr}\left(KFm^{+}M\right) \\ \qquad - \mathrm{Tr}\left(EM^{+}mFm^{+}M\right) - \mathrm{Tr}\left(Km^{+}MKm^{+}M\right) \\ \varepsilon_{yT}^{2} = |F_T| = |N|^{2}\varepsilon_y^{2} + |n|^{2}\varepsilon_x^{2} + 2|N||n|\kappa \\ \qquad - 2|n|\mathrm{Tr}\left(EKN^{+}n\right) - 2|N|\mathrm{Tr}\left(KFN^{+}n\right) \\ \qquad - \mathrm{Tr}\left(En^{+}NFN^{+}n\right) - \mathrm{Tr}\left(KN^{+}nKN^{+}n\right) \\ \kappa_T = |K_T| = |M||n|\varepsilon_x^{2} + |N||m|\varepsilon_y^{2} + \left(|M||N| + |m||n|\right)\kappa \\ \qquad - |M|\mathrm{Tr}\left(EKN^{+}n\right) - |m|\mathrm{Tr}\left(KFN^{+}n\right) \\ \qquad - |n|\mathrm{Tr}\left(EKm^{+}M\right) - |N|\mathrm{Tr}\left(KFm^{+}M\right) \\ \qquad - \mathrm{Tr}\left(En^{+}NFm^{+}M\right) - \mathrm{Tr}\left(KN^{+}nKm^{+}M\right) \end{cases}, \quad (33)$$

where we have defined $|K| \equiv \kappa$ as the "couplance." (Note that |K| is not always ≥ 0 as are |E| and |F|.)

If $\mathbf{M}$ is symplectic, Eqs. (33) are simplified by Eqs. (30) to

$$\begin{cases} \varepsilon_{xT}^{2} = |M|^{2}\varepsilon_x^{2} + |m|^{2}\varepsilon_y^{2} + 2|M||m|\kappa - 2|M|a - 2|m|b + c \\ \varepsilon_{yT}^{2} = |m|^{2}\varepsilon_x^{2} + |M|^{2}\varepsilon_y^{2} + 2|M||m|\kappa + 2|m|a + 2|M|b + c, \\ \kappa_T = \left(|M|^{2} + |m|^{2}\right)\kappa + |M||m|\left(\varepsilon_x^{2} + \varepsilon_y^{2}\right) + \left(|M| - |m|\right)\left(a - b\right) - c \end{cases} \quad (34)$$

where

$$\begin{cases} a \equiv \mathrm{Tr}\left(EKW\right), \qquad W \equiv m^{+}M \\ b \equiv \mathrm{Tr}\left(KFW\right) \\ c \equiv -\mathrm{Tr}\left(EW^{+}FW\right) - \mathrm{Tr}\left(KWKW\right) \end{cases}. \quad (35)$$

Equations (34) show directly

$$\varepsilon_{xT}^{2} + \varepsilon_{yT}^{2} + 2\kappa_T = \varepsilon_x^{2} + \varepsilon_y^{2} + 2\kappa\,, \quad (36)$$

a well-known invariant for symplectic transformations.

EXAMPLES OF APPLICATION

There are two classes of x-y coupled beam transports—the skew quadrupole train and the solenoid.

Class 1 Skew Quadrupole Train

This is a case of symplectic "deformation" propagation. The transfer matrix is

$$\mathbf{M} = \begin{pmatrix} I\cos\frac{\pi}{4} & -I\sin\frac{\pi}{4} \\ I\sin\frac{\pi}{4} & I\cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} I\cos\frac{\pi}{4} & I\sin\frac{\pi}{4} \\ -I\sin\frac{\pi}{4} & I\cos\frac{\pi}{4} \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} P+Q & P-Q \\ P-Q & P+Q \end{pmatrix}, \tag{37}$$

where P and Q are the principal 2-D transfer matrices along the quadrupole axes and the end matrices rotate the quadrupole train 45°. In this case

$$\begin{cases} |M| = \frac{1}{4}|P+Q| = \frac{1}{2}\left[1 + \frac{1}{2}\mathrm{Tr}\left(P^{+}Q\right)\right] \\ |m| = \frac{1}{4}|P-Q| = \frac{1}{2}\left[1 - \frac{1}{2}\mathrm{Tr}\left(P^{+}Q\right)\right] \\ W = m^{+}M = \frac{1}{4}\left(P^{+} - Q^{+}\right)\left(P+Q\right) = \frac{1}{4}\left(P^{+}Q - Q^{+}P\right) \end{cases}. \tag{38}$$

We parameterize the symplectic matrix $P^{+}Q$ as

$$P^{+}Q \equiv \cos 2\phi + J\sin 2\phi, \tag{39}$$

then

$$\begin{cases} |M| = \cos^{2}\phi, \qquad |m| = \sin^{2}\phi \\ W = m^{+}M = (\sin\phi\cos\phi)J \end{cases}. \tag{40}$$

Substituting Eqs. (40) in Eqs. (34) we get the transformed emittances.

Class 2A Solenoid – Whole

The transport through a whole solenoid (from exterior to exterior where the vector potential is zero) is a symplectic "rotation." The transfer matrix is

$$\mathbf{M} = \begin{pmatrix} R\cos\psi & R\sin\psi \\ -R\sin\psi & R\cos\psi \end{pmatrix} \tag{41}$$

with the 2-D

$$R \equiv \begin{pmatrix} \cos\psi & \frac{1}{k}\sin\psi \\ -k\sin\psi & \cos\psi \end{pmatrix}, \tag{42}$$

where

$$\psi \equiv \left(\frac{1}{2}\frac{B_z}{B\rho}\right)\ell \equiv k\ell,$$

and B_z and ℓ are the solenoid field and length and $B\rho$ is the rigidity of the beam. This gives

$$\begin{cases} |M| = \cos^{2}\psi, \qquad |m| = \sin^{2}\psi \\ W = m^{+}M = (\sin\psi\cos\psi)\, I \end{cases}. \tag{43}$$

Compared to Eqs. (40) we see that J is here replaced by I. This distinguishes the "deformation" of a skew-quadrupole train from the "rotation" of a solenoid.

Class 2B Solenoid – Ends

The only easily available nonsymplectic case is when either (or both) end of the transport is in the interior of the solenoid where the transverse vector potential $A_{\perp} \neq 0$. We give here the thin "entry" of a solenoid. The transfer matrix is

$$M = N = I \qquad m = -m^{+} = n^{+} = -n = \begin{pmatrix} 0 & 0 \\ k & 0 \end{pmatrix}. \tag{44}$$

In this case we have to go back to the general (nonsymplectic) formulas in Eqs. (33). We have |M| = |N| = 1 and |m| = |n| = 0. For the emittances Eqs. (33) give

$$\begin{cases} \varepsilon_{xT}^{2} = \varepsilon_{x}^{2} + 2a + c \\ \varepsilon_{yT}^{2} = \varepsilon_{y}^{2} + 2b + c, \\ \kappa_{T} = \kappa + a + b + c \end{cases} \tag{45}$$

where

$$\begin{cases} a = -\mathrm{Tr}\left(EKm^{+}M\right) = k\left(\overline{x^{2}}\,\overline{x'y} - \overline{xy}\,\overline{xx'}\right) \\ b = -\mathrm{Tr}\left(KFm^{+}M\right) = -k\left(\overline{y^{2}}\,\overline{xy'} - \overline{xy}\,\overline{yy'}\right) \\ c = -\mathrm{Tr}\left(EM^{+}mFm^{+}M\right) - \mathrm{Tr}\left(Km^{+}MKm^{+}M\right) \\ \quad = k^{2}\left(\overline{x^{2}}\,\overline{y^{2}} - \overline{xy}^{2}\right) \end{cases}. \tag{46}$$

This shows the noninvariance of

$$\varepsilon_{xT}^{2} + \varepsilon_{yT}^{2} + 2\kappa_{T} = \varepsilon_{x}^{2} + \varepsilon_{y}^{2} + 2\kappa + 4(a+b+c). \tag{47}$$

The "exit" is identical to the "entry" except with the sign of k reversed.

With these three classes of transport one should be able to obtain most desired emittance transformations. If, indeed, there is a transformation that cannot be obtained from combinations of these cases (plus uncoupled transports), some yet unknown coupled transports must be invented.

THREE DEGREES OF FREEDOM

For 3-dof the phase-point position is given by a 1×3 block vector. The second-moment matrix and the transfer matrix are both 3×3=9 block matrices. The inverse second-moment matrix and the conditions for symplecticity are more complex than the 2-dof case, but can be given in a straightforward manner.