Proceedings of the 2001 Particle Accelerator Conference, Chicago

ANALYTICAL ESTIMATION OF THE DYNAMIC APERTURES OF CIRCULAR ACCELERATORS

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Abstract

By considering delta function multipole perturbations and using difference action-angle variable equations, we find some useful analytical formulae for the estimation of the corresponding dynamic apertures of circular accelerators based on the Chirikov criterion of the onset of stochastic motions. Their combined effects and two dimensional dynamic apertures are discussed also. Comparisons with numerical simulations are made, and the agreement is quite satisfactory.

1 INTRODUCTION

In this paper we will show how single sextupole, and single $2m$ pole in general in a storage ring limit the dynamic aperture and what is their combined effect if there are more than one nonlinear element. From the established analytical formulae for the dynamic aperture one gets the scaling laws which relate the nonlinear perturbation strengths, beta functions, and the dynamic apertures. Restricted to sextupole case, we establish the corresponding 2D analytical dynamic aperture expression.

2 ANALYTICAL FORMULAE FOR DYNAMIC APERTURES

To start with we consider the linear horizontal motion of the reference particle (no energy deviation) in the horizontal plane (y=0) assuming that the magnetic field is only transverse ($A_x = A_y = 0$) and has no skew fields, and Φ is a constant. The Hamiltonian can be simplified as

$$H = \frac{p^2}{2} + \frac{K(s)}{2}x^2 \tag{1}$$

where x denotes normal plane coordinate, $p = dx/ds$, and $K(s)$ is a periodic function satisfying the relation

$$K(s) = K(s+L) \tag{2}$$

where L is the circumference of the ring. The solution of the deviation, x, is found to be

$$x = \sqrt{\epsilon_x \beta_x(s)}\cos(\phi(s) + \phi_0) \tag{3}$$

where

$$\phi(s) = \int_0^s \frac{ds}{\beta_x(s)} \tag{4}$$

As an essential step towards further discussion on the motions under nonlinear perturbation forces, we introduce action-angle variables and the Hamiltonian expressed in these new variables:

$$\Psi = \int_0^s \frac{ds'}{\beta_x(s')} + \phi_0 \tag{5}$$

$$J = \frac{\epsilon_x}{2} = \frac{1}{2\beta_x(s)}\left(x^2 + \left(\beta_x(s)x' - \frac{\beta_x' x}{2}\right)^2\right) \tag{6}$$

$$H(J,\Psi) = \frac{J}{\beta_x(s)} \tag{7}$$

Since $H(J,\Psi) = J/\beta_x(s)$ is still a function of the independent variable, s, we will make another canonical transformation to freeze the new Hamiltonian:

$$\Psi_1 = \Psi + \frac{2\pi\nu}{L} - \int_0^s \frac{ds'}{\beta_x(s')} \tag{8}$$

$$J_1 = J \tag{9}$$

$$H_1 = \frac{2\pi\nu}{L}J_1 \tag{10}$$

Before going on further, let's remember the relation between the last action-angle variables and the particle deviation x:

$$x = \sqrt{2J_1\beta_x(s)}\cos\left(\Psi_1 - \frac{2\pi\nu}{L}s + \int_0^s \frac{ds'}{\beta_x(s')}\right) \tag{11}$$

Being well prepared, we start our journey to find out the limitations of the nonlinear forces on the stability of the particle's motion. Limited by the space we consider at this stage only a sextupole (no skew term). The perturbed one dimensional Hamiltonian can thus be expressed:

$$H = \frac{p^2}{2} + \frac{K(s)}{2}x^2 + \frac{1}{3!B\rho}\frac{\partial^2 B_z}{\partial x^2}x^3 L\sum_{k=-\infty}^{\infty}\delta(s-kL) \tag{12}$$

Representing eq. 12 by action-angle variables (J_1 and Ψ_1), and using

$$B_z = B_0(1 + xb_1 + x^2 b_2) \tag{13}$$

one has

$$H = \frac{2\pi\nu}{L}J_1 + \frac{(2J_1\beta_x(s_1))^{3/2}}{3\rho}b_2 L\cos^3\Psi_1\sum_{k=-\infty}^{\infty}\delta(s-kL) \tag{14}$$

where s_1 and s_2 are just used to differentiate the locations of the sextupole and the octupole perturbations. By virtue of Hamiltonian one gets the differential equations for Ψ_1 and J_1

$$\frac{dJ_1}{ds} = -\frac{\partial H_1}{\partial \Psi_1} \tag{15}$$

$$\frac{d\Psi_1}{ds} = \frac{\partial H_1}{\partial J_1} \tag{16}$$

$$\frac{dJ_1}{ds} = -\frac{(2J_1\beta_x(s_1))^{3/2}}{3\rho}b_2 L\frac{d\cos^3\Psi_1}{d\Psi_1}\sum_{k=-\infty}^{\infty}\delta(s-kL) \tag{17}$$

$$\frac{d\Psi_1}{ds} = \frac{2\pi\nu}{L} + \frac{\sqrt{2}J_1^{1/2}\beta_x(s_1)^{3/2}}{\rho} b_2 L \cos^3\Psi_1 \sum_{k=-\infty}^{\infty} \delta(s-kL) \tag{18}$$

We now change this differential equations to the difference equations which are *suitable* to analyse the possibilities of the onset o stochasticity [2][3]. Since the perturbations have a natural periodicity of L we will sample the dynamic quantities at a sequence of s_i with constant interval L assuming that the characteristic time between two consecutive adiabatic invariance breakdown intervals is shorter than L/c. The differential equations in eqs. 17 and 18 are reduced to

$$\overline{J_1} = \overline{J_1}(\Psi_1, J_1) \tag{19}$$

$$\overline{\Psi_1} = \overline{\Psi_1}(\Psi_1, J_1) \tag{20}$$

where the bar stands for the next sampled value after the corresponding unbared previous value.

$$\overline{J_1} = J_1 - \frac{(2J_1\beta_x(s_1))^{3/2}}{3\rho} b_2 L \frac{d\cos^3\Psi_1}{d\Psi_1} \tag{21}$$

$$\overline{\Psi_1} = \Psi_1 + 2\pi\nu + \frac{\sqrt{2}\beta_x(s_1)^{3/2}\overline{J_1}^{1/2}}{\rho} b_2 L \cos^3\Psi_1 \tag{22}$$

Eqs. 21 and 22 are the basic difference equations to study the nonlinear resonance and the onset of stochasticities considering sextupole and octupole perturbations. By using trigonometric relation

$$\cos^m\theta\cos n\theta = 2^{-m}\sum_{r=0}^{m} \frac{m!}{(m-r)!r!}\cos(n-m+2r)\theta \tag{23}$$

one has

$$\cos^3\theta = \frac{2}{2^3}(\cos 3\theta + 3\cos\theta) \tag{24}$$

$$\cos^4\theta = \frac{1}{2^4}(2\cos 4\theta + 8\cos 2\theta + \frac{4!}{((4/2)!)^2}) \tag{25}$$

If the tune ν is far from the resonance lines $\nu = m/n$, where m and n are integers, the invariant tori of the unperturbed motion are preserved under the presence of the small perturbations by virtue of the Kolmogorov-Arnold-Moser (KAM) theorem. If, however, ν is close to the above mentioned resonance line, under some conditions the KAM invariant tori can be broken.

Consider now the case where there is only one sextupole located at $s = s_1$ with $\beta_x(s_1)$. Taking the third order resonance, $m/3$, for example, we keep only the sinusoidal function with phase $3\Psi_1$ in eq. 21 and the dominant phase independent nonlinear term in eq. 22, and as the result, eqs. 21 and 22 become

$$\overline{J_1} = J_1 + A\sin 3\Psi_1 \tag{26}$$

$$\overline{\Psi_1} = \Psi_1 + B\overline{J_1} \tag{27}$$

with

$$A = \frac{(2J_1\beta_x(s_1))^{3/2}}{4}\left(\frac{b_2 L}{\rho}\right) \tag{28}$$

$$B = \sqrt{2}\beta_x(s_1)^{3/2}J_1^{-1/2}\left(\frac{b_2 L}{\rho}\right) \tag{29}$$

where we have dropped the constant phase in eq. 22 and take the maximum value of $\cos^3(\Psi_1)$, 1. It is helpful to transform eqs. 28 and 29 into the form so-called *standard mapping* [3] expressed as

$$\overline{I} = I + K_0\sin\theta \tag{30}$$

$$\overline{\theta} = \theta + \overline{I} \tag{31}$$

with $\theta = 3\Psi$, $I = 3BJ_1$ and $K_0 = 3AB$. By virtue of the Chirikov criterion [3] it is known that when $|K_0| \geq 0.97164$ [4] resonance overlapping occurs which results in particles' stochastic motions and diffusion processes. Therefore,

$$|K_0| \leq 1 \tag{32}$$

can be taken as a natural criterion for the determination of the dynamic aperture of the machine. Putting eqs. 28 and 29 into eq. 32, one gets

$$|K_0| = 3J_1\beta_x(s_1)^3\left(\frac{|b_2|L}{\rho}\right)^2 \leq 1 \tag{33}$$

and consequently, one finds maximum J_1 corresponding to a $m/3$ resonance

$$J_1 \leq J_{max,sext} = \frac{1}{3\beta_x(s_1)^3}\left(\frac{\rho}{|b_2|L}\right)^2 \tag{34}$$

The dynamic aperture of the machine is therefore

$$A_{dyna,sext} = \sqrt{2J_{max,sext}\beta_x(s)} = \frac{\sqrt{2\beta_x(s)}}{\sqrt{3}\beta_x(s_1)^{3/2}}\left(\frac{\rho}{|b_2|L}\right) \tag{35}$$

Eq. 35 gives the dynamic aperture of a sextuple strength determined case. The reader can confirm that if we keep $\sin(\Psi_1)$ term instead of $\sin(3\Psi_1)$ in eq. 26, one arrives at the same expression for $A_{dyna,sext}$ as expressed in eq. 35.

Finally, we give the general expression of the dynamic aperture in the horizontal plane ($z = 0$) of a single $2m$ ($m \geq 3$) pole component:

$$A_{dyna,2m} = \sqrt{2\beta_x(s)}\left(\frac{1}{m\beta_x^m(s(2m))}\right)^{\frac{1}{2(m-2)}} \times \left(\frac{\rho}{|b_{m-1}|L}\right)^{1/(m-2)} \tag{36}$$

where $s(2m)$ is the location of this multipole. Eq. 36 gives us useful scaling laws, such as $A_{dyna,2m} \propto \left(\frac{\rho}{|b_{m-1}|L}\right)^{1/(m-2)}$, and $A_{dyna,2m} \propto \left(\frac{1}{\beta_x^m(s(2m))}\right)^{\frac{1}{2(m-2)}}$.

If there is more than one nonlinear component, how can one estimate their collective effect? Fortunately, one can distinguish two cases:

1) If the components are *independent*, i.e. there are no special phase and amplitude relations between them, the total dynamic aperture can be calculated as:

$$\frac{1}{A_{dyna,total}^2} = \sum_i \frac{1}{A_{dyna,sext,i}^2} + \sum_j \frac{1}{A_{dyna,oct,j}^2} + \cdots \quad (37)$$

2) If the nonlinear components are *dependent*, i.e. there are special phase and amplitude relations between them (for example, in reality, one use some additional sextupoles to cancel the nonlinear effects of the sextupoles used to make chromaticity corrections), there is no general formula as eq. 37 to apply.

In the above discussion we have restricted us to the case where particles are moving in the horizontal plane, and the one dimensional dynamic aperture formulae expressed in eq. 37 are the maximum stable horizontal excursion ranges with the vertical displacement $y = 0$. In the following we will show briefly how to estimate the dynamic aperture in 2 dimensions when there is coupling between the horizontal and vertical planes. Now we consider the case where only one sextupole is located at $s = s_1$, and we have the corresponding Hamiltonian expressed as follows:

$$H = \frac{p_x^2}{2} + \frac{K_x(s)}{2}x^2 + \frac{p_y^2}{2} + \frac{K_y(s)}{2}y^2$$

$$+\frac{1}{3!B\rho}\frac{\partial^2 B_z}{\partial x^2}(x^3 - 3xy^2)L\sum_{k=-\infty}^{\infty}\delta(s-kL) \quad (38)$$

Generally speaking, there exists no universal criterion to determine the start up of stochastic motions in 2D. Fortunately, in our specific case, we find out the similarity between the Hamiltonian expressed in eq. 38 and that of the Hénon and Heiles problem which has been much studied in literature [5]. The Hénon and Heiles problem's Hamiltonian is given by

$$H_{H\&H} = \frac{1}{2}\left(x^2 + p_x^2 + y^2 + p_y^2 + 2y^2x - \frac{2}{3}x^3\right) \quad (39)$$

When $H_{H\&H} > 1/6$ the motion becomes unstable. The intuition we get from this conclusion is that there should exist a similar criterion for our problem, i.e. to have stable 2D motion one should have $H \leq H_{max}$. Note that $K_x(s)$ and $K_y(s)$ in eq. 38 are equal to unity in the Hénon and Heiles problem's Hamiltonian. The previous one dimensional result helps us now to find H_{max}. When $y = 0$ one has $H_{max} \propto A_{dyna,sext,x}^2$, since $x \leq A_{dyna,sext,x}$. When $y \neq 0$, the crossing terms in eqs. 38 and 39 will play the role of exchanging energy between the two planes, and for a given set of x and y the total energy of the coupled system can not exceed H_{max}. If we define $A_{dyna,sext,y}$ is the dynamic aperture in y-plane, one has:

$$\beta_x(s_1)A_{dyna,sext,x}^2 = \beta_y(s_1)A_{dyna,sext,y}^2 + \beta_x(s_1)x^2 \quad (40)$$

or:

$$A_{dyna,sext,y} = \sqrt{\frac{\beta_x(s_1)}{\beta_y(s_1)}(A_{dyna,sext,x}^2 - x^2)} \quad (41)$$

where $\beta_y(s_1)$ is the vertical beta function where the sextupole is located and $A_{dyna,sext,x}$ is given by eq. 35. Different from eq. 36, the derivation of eq. 41 is quite intuitive, hinted by the Hénon and Heiles problem which has been studied numerically instead of analytically in literature. From eq. 41 one understands that the difference between $A_{dyna,sext,y}$ and $A_{dyna,sext,x}$ comes from $\sqrt{\beta_x(s_1)/\beta_y(s_1)}$. If there are many sextupoles in a ring one usually has $A_{dyna,sext,x} \approx A_{dyna,sext,y}$ since $\beta_x(s_i)$ will not be always larger or smaller than $\beta_y(s_i)$.

To check the validities of eqs. 36, 37, and 41 we have used the lattice of Super-ACO as a test band and used a program named BETA [6] as the numerical simulation tool. After a systematic comparison it is found that these analytical expressions are valid. The detailed comparison results can be found in ref. [7].

3 CONCLUSION

We have derived the analytical formulae for the dynamic apertures in circular accelerators due to single $2m$ pole in general, and the combination of many independent multipoles. A very interesting application of these analytical formulae is to calculate the beam-beam limited dynamic aperture and lifetime as shown in detail in ref. [8].

4 ACKNOWLEDGEMENTS

The author of this paper thanks B. Mouton for his help in using BETA program and also for his generating a flexible lattice based on the original Super-ACO one. The fruitful discussions with A. Tkachenko are very much appreciated.

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