EFFECTS OF TRANSVERSE BEAM SIZE IN BEAM POSITION MONITORS*

Sergey S. Kurennoy, LANL, Los Alamos, NM 87545, USA

Abstract

The fields produced by a long beam with a given transverse charge distribution in a homogeneous vacuum chamber are studied. Signals induced by the displaced finite-size beam on electrodes of a beam position monitor (BPM) are calculated and compared to those from a pencil beam. The non-linearities and corrections to BPM signals due to a finite transverse beam size are calculated for an arbitrary chamber cross section. Simple analytical expressions are given for a few particular transverse distributions of the beam current in a circular or rectangular chamber. Of particular interest is a general proof that in an arbitrary homogeneous chamber the beam-size corrections vanish for any axisymmetric beam current distribution.

1 INTRODUCTION

In many ion linacs and storage rings beams occupy a significant fraction of the vacuum chamber cross section. On the other hand, an analysis of beam-induced signals in beam position monitors (BPMs) is usually restricted to the case of an infinitely small beam cross section (pencil beam). Here we calculate, for a given transverse charge distribution of an off-axis beam and for a vacuum chamber with an arbitrary but constant cross section, the field produced by the beam on the chamber wall. Comparing it with the field of a pencil beam gives us corrections due to a finite transverse size of the beam.

Let a vacuum chamber have an arbitrary cross section S that does not change as a beam moves along the chamber axis z, and perfectly conducting walls. We consider the case of $(\omega b/\beta\gamma c)^2 \ll 1$, where ω is the frequency of interest, βc is the beam velocity, $\gamma = 1/\sqrt{1-\beta^2}$, and b is a typical transverse dimension of the vacuum chamber. It includes both the ultra relativistic limit, $\gamma \gg 1$, and the long-wavelength limit when, for a fixed γ, the wavelength of interest $\lambda \gg 2\pi b/\gamma$. Under these assumptions the problem of calculating the beam fields at the chamber walls is reduced to a 2-D electrostatic problem of finding the field of the transverse distribution $\lambda(\vec{r})$ of the beam charge, which occupies region S_b of the beam cross section, on the boundary of region S, see [1]. Let the beam transverse charge distribution $\lambda(\vec{r})$ satisfy $\int_{S_b} d\vec{r}\lambda(\vec{r}) = 1$, which means the unit charge per unit length of the beam. If we know the field $e(\vec{r},\vec{b})$ produced at a point $\vec{b}$ on the wall by a pencil beam located at a point $\vec{r}$ of S, the thick-beam field is

$$E(\vec{a},\vec{b}) = \int_{S_b} d\vec{r}\lambda(\vec{r})e(\vec{r},\vec{b}) \; , \tag{1}$$

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where vector $\vec{a}$ is defined as the center of the charge distribution: $\vec{a} = \int d\vec{r}\vec{r}\lambda(\vec{r})$. We will denote $\tilde{\lambda}(\vec{r'}) \equiv \lambda(\vec{a}+\vec{r'})$, where the tilde in $\tilde{\lambda}$ means that the argument of the distribution function λ is shifted so that the vector $\vec{r'}$ now originates from the beam center. Let us start with a particular case of a circular cylindrical vacuum chamber.

2 CIRCULAR CHAMBER

In a circular cylindrical pipe of radius b, a pencil beam with a transverse offset $\vec{r}$ from the axis produces the following electric field on the wall

$$e(\vec{r},\vec{b}) = \frac{1}{2\pi b}\frac{b^2-r^2}{b^2-2br\cos(\theta-\varphi)+r^2} = \tag{2}$$
$$\frac{1}{2\pi b}\left\{1+2\sum_{k=1}^{\infty}\left(\frac{r}{b}\right)^k \cos\left[k(\theta-\varphi)\right]\right\},$$

where φ, θ are the azimuthal angles of vectors $\vec{r}, \vec{b}$, correspondingly. Integrating the multipole expansion (2) with a double-Gaussian distribution of the beam charge

$$\tilde{\lambda}(x,y) = \exp\left(-x^2/2\sigma_x^2 - y^2/2\sigma_y^2\right)/\left(2\pi\sigma_x\sigma_y\right) \; , \tag{3}$$

(assuming, of course, the rms beam sizes $\sigma_x, \sigma_y \ll b$), one obtains non-linearities in the form of powers of a_x, a_y, as well as the beam size corrections, which come as powers of σ_x, σ_y. To our knowledge, this was done first for a circular pipe by R.H. Miller *et al* in the 1983 paper [2], where the expansion was calculated up to the 3rd order terms. Recently their results have been used at LANL to calculate the beam emittance from the second-order beam moments measured by BPMs [3]. In a recent series of papers [4] by CERN authors, the results [2] have been recalculated (and corrected in the 3rd order), and applied for beam-size measurements with movable BPMs.

In fact, integrating (2) with the distribution (3) can be readily carried out up to an arbitrary order, see [5]. The result, up to the 5th order terms, looks as follows

$$\begin{aligned} E(\vec{r}_0,\vec{b}) &= \frac{1}{2\pi b} + \frac{1}{\pi b^2}\left\{\cos\theta\; x_0 + \sin\theta\; y_0\right\} \\ &+ \frac{1}{\pi b^3}\left\{\cos 2\theta\;\left(\sigma_x^2-\sigma_y^2+x_0^2-y_0^2\right) + \sin 2\theta\; 2x_0y_0\right\} \\ &+ \frac{1}{\pi b^4}\left\{\begin{array}{l}\cos 3\theta\; x_0\left[3\left(\sigma_x^2-\sigma_y^2\right)+x_0^2-3y_0^2\right] + \\ \sin 3\theta\; y_0\left[3\left(\sigma_x^2-\sigma_y^2\right)+3x_0^2-y_0^2\right]\end{array}\right\} \\ &+ \frac{1}{\pi b^5}\left\{\begin{array}{l}\cos 4\theta\;\left[\begin{array}{c}3\left(\sigma_x^2-\sigma_y^2+x_0^2-y_0^2\right)^2\\ -2x_0^4-2y_0^4\end{array}\right] + \\ \sin 4\theta\; 4x_0y_0\left[3\left(\sigma_x^2-\sigma_y^2\right)+x_0^2-y_0^2\right]\end{array}\right\} \end{aligned} \tag{4}$$

$$+\frac{1}{\pi b^6}\left\{\begin{array}{l}\cos 5\theta\ x_0\left[\begin{array}{l}15\left(\sigma_x^2-\sigma_y^2\right)^2+\\10\left(\sigma_x^2-\sigma_y^2\right)\left(x_0^2-3y_0^2\right)\\+x_0^4-10x_0^2y_0^2+5y_0^4\end{array}\right]+\\\sin 5\theta\ y_0\left[\begin{array}{l}15\left(\sigma_x^2-\sigma_y^2\right)^2+\\10\left(\sigma_x^2-\sigma_y^2\right)\left(3x_0^2-y_0^2\right)\\+5x_0^4-10x_0^2y_0^2+y_0^4\end{array}\right]\end{array}\right\}$$
$$+\ldots$$

The expansion (4) that includes terms up to the decapole ones, leads us to one interesting observation: all beam-size corrections come in the form of the difference $\sigma_x^2-\sigma_y^2$, and vanish for a round beam where $\sigma_x^2=\sigma_y^2$. It was shown in [5] by directly integrating the field (2) in Eq. (1) with an arbitrary azimuthally symmetric distribution of the beam charge $\tilde\lambda(\vec r)=\tilde\lambda(r)$ that the beam-size corrections in a circular pipe vanish in all orders, and the resulting field is exactly equal to that of a pencil beam with the transverse position at the center of the finite-size beam.

Consider now a stripline BPM with subtended angle φ per stripline and stripline electrodes flush with the circular chamber of radius b. Integrating the field (4) on the electrodes, one obtains the ratio of the difference between the signals on the right (*R*) and left (*L*) electrodes in the horizontal plane to the sum of these signals:

$$\begin{aligned}\frac{R-L}{R+L}=2\frac{x_0}{b}\frac{\sin\varphi/2}{\varphi/2}\times\Big\{1+O\left(b^{-6}\right)-\\
-\frac{2}{b^2}\frac{\sin\varphi}{\varphi}\left(\sigma_x^2-\sigma_y^2+x_0^2-y_0^2\right)\\
+\frac{1}{b^2}\frac{\sin 3\varphi/2}{3\varphi/2}\left(\sigma_x^2-\sigma_y^2+x_0^2/3-y_0^2\right)\\
-\frac{2}{b^4}\frac{\sin 2\varphi}{2\varphi}\left[\left(\sigma_x^2-\sigma_y^2+x_0^2-y_0^2\right)^2-2x_0^4-2y_0^4\right]\\
+\frac{4}{b^4}\left(\frac{\sin\varphi}{\varphi}\right)^2\left(\sigma_x^2-\sigma_y^2+x_0^2-y_0^2\right)^2\\
+\frac{1}{b^4}\frac{\sin 5\varphi/2}{5\varphi/2}\left[\begin{array}{l}3\left(\sigma_x^2-\sigma_y^2\right)^2\\+2\left(\sigma_x^2-\sigma_y^2\right)\left(x_0^2-3y_0^2\right)\\+x_0^4/5-2x_0^2y_0^2+y_0^4\end{array}\right]\Big\}\end{aligned}\tag{5}$$

The factor outside $\{\ldots\}$ in the RHS of Eq. (5) is the linear part of the BPM response, so that all terms in $\{\ldots\}$ except 1 give non-linearities and beam-size corrections. Corrections (5) have been plotted for a 60° stripline BPM in [5]. For a reasonable transverse beam size the beam-size corrections are on the level of a few percents.

3 ARBITRARY CROSS SECTION

For the general case of a homogeneous vacuum chamber with an arbitrary single-connected cross section S, the field $e(\vec r,\vec b)$ produced by a pencil beam at a point $\vec b$ on the wall can be written as (e.g. see [6])

$$e(\vec r,\vec b)=-\sum\nolimits_s k_s^{-2}e_s(\vec r)\nabla_\nu e_s(\vec b)\,, \tag{6}$$

where $s=(n,m)$ is a generalized (2-D) index, $\nabla_\nu=\vec\nabla\cdot\vec\nu$ is a normal derivative at the point $\vec b$ on the region boundary ∂S, and $k_s^2, e_s(\vec r)$ are eigenvalues and orthonormalized eigenfunctions of a 2-D Dirichlet problem in S:

$$\left(\nabla^2+k_s^2\right)e_s(\vec r)=0;\quad e_s(\vec r\in\partial S)=0\,. \tag{7}$$

The eigenfunctions for simple regions can be found in electrodynamics textbooks, or in [6]. For the circular case, summing the Bessel functions in (6) leads directly to Eq. (2). For a thick beam with a given transverse charge distribution, from Eqs. (1) and (6)

$$E(\vec a,\vec b)=-\sum\nolimits_s k_s^{-2}\nabla_\nu e_s(\vec b)\int_{S_b}\tilde\lambda(\vec r)e_s(\vec a+\vec r)d\vec r. \tag{8}$$

Performing the Taylor expansion of $e_s(\vec a+\vec r)$ around point $\vec a$, and integrating in (8) leads to a series in multipoles of the beam charge distribution. It was shown in [5] that for symmetric charge distributions, i.e. when $\tilde\lambda(-\vec r)=\tilde\lambda(\vec r)$, the expansion takes a simple form:

$$E(\vec a,\vec b)=e(\vec a,\vec b)+\sum_{n=1}^{\infty}\partial_x^{2n}e(\vec a,\vec b)\ M_{2n}/\left(2n\right)! \tag{9}$$

where $\partial_x\equiv\partial/\partial x$, and in the polar coordinates of the beam

$$M_{2n}=\int_{S_b}d\vec r\tilde\lambda(\vec r)\ r^{2n}\cos 2n\varphi\,. \tag{10}$$

It is clear that for any azimuthally symmetric distribution of the beam charge $\tilde\lambda(\vec r)=\tilde\lambda(r)$, i.e. $\tilde\lambda(r,\varphi)=\tilde\lambda(r)$, all beam-size corrections in (9) vanish after the angular integration. Therefore, a rather general statement is proved: *the fields produced by a beam with an azimuthally-symmetric charge distribution on the walls of a homogeneous vacuum chamber of an arbitrary cross section are exactly the same as those due to a pencil beam of the same current following the same path.*

Expansion (9) gives the beam-size corrections for symmetric beam-charge distributions in an arbitrary chamber. One just has to calculate the even moments of the distribution, $M_2=\int_{S_b}d\vec r\tilde\lambda(\vec r)\left(x^2-y^2\right)$, $M_4=\int_{S_b}d\vec r\tilde\lambda(\vec r)\left(x^4-6x^2y^2+y^4\right)$, etc. For the double Gaussian beam (3), $M_2=\sigma_x^2-\sigma_y^2$, $M_4=3\left(\sigma_x^2-\sigma_y^2\right)^2$, etc., so that from Eq. (9) follows

$$\begin{aligned}E(\vec a,\vec b)&=e(\vec a,\vec b)+\frac{1}{2}\left(\sigma_x^2-\sigma_y^2\right)\partial_x^2e(\vec a,\vec b)\\
&+\frac{1}{8}\left(\sigma_x^2-\sigma_y^2\right)^2\partial_x^4e(\vec a,\vec b)\\
&+\frac{1}{48}\left(\sigma_x^2-\sigma_y^2\right)^3\partial_x^6e(\vec a,\vec b)+O(\sigma^8)\,.\end{aligned}\tag{11}$$

For a uniform beam with the rectangular cross section $2\sigma_x\times 2\sigma_y$, the corrections are

$$\begin{aligned}E(\vec a,\vec b)=e(\vec a,\vec b)+\frac{1}{6}\left(\sigma_x^2-\sigma_y^2\right)\partial_x^2e(\vec a,\vec b)+O(\sigma^8)\\
+\frac{1}{40}\left(\sigma_x^4-\frac{10}{3}\sigma_x^2\sigma_y^2+\sigma_y^4\right)\partial_x^4e(\vec a,\vec b)\\
+\frac{1}{5040}\left(\sigma_x^6-7\sigma_x^4\sigma_y^2+7\sigma_x^2\sigma_y^4-\sigma_y^6\right)\partial_x^6e(\vec a,\vec b)\,.\end{aligned}\tag{12}$$

For a round beam, $\sigma_x = \sigma_y$, all corrections in (11) disappear as expected, while for a square beam in (12), the lowest correction is $\propto \sigma^4$, and the next one $\propto \sigma^8$.

A general expansion for both the non-linearities and beam-size corrections was derived by expanding in Eq. (9) the pencil-beam field $e(\vec{a}, \vec{b})$ and its derivatives in powers of a , see [5] for detail. Let region S be symmetric with respect to its vertical and horizontal axis, and a pair of narrow BPM electrodes be placed on the walls in the horizontal plane of such chamber. Using symmetry, the difference over sum signal ratio of BPM signals can be expressed as

$$\frac{R-L}{R+L} = \frac{x_0 \partial_x e_0}{e_0} \times \left\{1 + O\left(r_0^6/b^6, \sigma^6/b^6\right) + \right. \tag{13}$$
$$+ \frac{1}{2}\frac{\partial_x^3 e_0}{\partial_x e_0}\left(\frac{x_0^2}{3} - y_0^2 + M_2\right) - \frac{1}{2}\frac{\partial_x^2 e_0}{e_0}\left(x_0^2 - y_0^2 + M_2\right)$$
$$+ \frac{1}{24}\frac{\partial_x^5 e_0}{\partial_x e_0}\left[\frac{x_0^4}{5} - 2x_0^2 y_0^2 + y_0^4 + 2M_2\left(x_0^2 - 3y_0^2\right) + M_4\right]$$
$$\left. - \frac{1}{24}\frac{\partial_x^4 e_0}{e_0}\left[x_0^4 - 6x_0^2 y^2 + y_0^4 + 6M_2\left(x_0^2 - y_0^2\right) + M_4\right]\right\},$$

where the notation $e_0 = e(0, \vec{b})$ was used for brevity. The factor in the RHS is the BPM linear response. The non-linearities are shown explicitly as powers of x_0 and y_0, and beam-size corrections enter via the moments of the beam charge distribution, cf. Eq. (9).

4 RECTANGULAR CHAMBER

Here we list some results for the particular case of a vacuum chamber with the rectangular cross section $w \times h$. For $-w/2 \le x_0 \le w/2$, $-h/2 \le y_0 \le h/2$, and for the particular charge distributions considered above, the thick-beam field (1) at point $\vec{b} = (w/2, y_h)$ on the right-side wall is

$$E(\vec{r}_0, \vec{b}) = \sum_{m=1}^{\infty} \sin \pi m \left(\frac{h + y_0}{2h}\right) \sin \pi m \left(\frac{h + y_h}{2h}\right) \tag{14}$$
$$\times \frac{2 \sinh \pi m \left(\frac{w}{2h} + \frac{x_0}{h}\right)}{h \sinh\left(\pi m w/h\right)} f\left(\frac{\pi m \sigma_y}{h}\right) F\left(\frac{\pi m \sigma_x}{w}\right).$$

The pencil-beam field is given by (14) with the form-factors $f(z) = F(z) \equiv 1$. For the double Gaussian beam (3), they are $f(z) = \exp(-z^2/2)$, $F(z) = \exp(z^2/2)$, and the corrections vanish when $\sigma_x = \sigma_y$. For the uniform rectangular distribution, the form-factors are $f(z) = \sin(z)/z$, $F(z) = \sinh(z)/z$, so that the corrections start from σ^4, as we already know.

For BPM signals, from Eq. (13) for two stripline BPM electrodes of width h_1 on side walls, the difference over sum signal ratio, up to the 5th order, is

$$\frac{R-L}{R+L} = \pi \frac{x_0}{h}\frac{\Sigma_1}{\Sigma_0} \times \left\{1 + O\left(r_0^6/b^6, \sigma^6/b^6\right) + \right. \tag{15}$$
$$\frac{\pi^2}{2h^2}\frac{\Sigma_3}{\Sigma_1}\left(\frac{x_0^2}{3} - y_0^2 + M_2\right) - \frac{\pi^2}{2h^2}\frac{\Sigma_2}{\Sigma_0}\left(x_0^2 - y_0^2 + M_2\right) +$$
$$\frac{\pi^4}{24h^4}\frac{\Sigma_5}{\Sigma_1}\left[\frac{x_0^4}{5} - 2x_0^2 y_0^2 + y_0^4 + 2M_2\left(x_0^2 - 3y_0^2\right) + M_4\right]$$
$$\left. - \frac{\pi^4}{24h^4}\frac{\Sigma_4}{\Sigma_0}\left[x_0^4 - 6x_0^2 y_0^2 + y_0^4 + 6M_2\left(x_0^2 - y_0^2\right) + M_4\right]\right\},$$

where

$$\Sigma_m = \sum_{k=0}^{\infty} (2k+1)^m \frac{\Phi\left(\pi\left(k + 1/2\right) h_1/h\right)}{\text{hyp}\left[\pi\left(k + 1/2\right) w/h\right]}$$

for $m = 0, 1, 2, \ldots$, and hyp $= \cosh$ or $\sinh$ for even or odd ms. The sums above include one more form-factor, $\Phi(z) = \sin(z)/z$, that accounts for the BPM electrode width. For narrow electrodes, when $h_1 \ll h$, $\Phi(z) \to 1$.

Corrections (15) have been plotted in [5] for a square chamber, $w = h$, and a BPM with very narrow electrodes. In this case, the non-linearities turn out to be practically the same for three different vertical beam offsets. On the contrary, the beam-size corrections depend noticeably on the beam vertical offset, and range from about +3% for $y_0 = 0$ (the chamber mid-plane) to less than 1% for $y_0 = h/8$ to about -(9-12)% for $y_0 = h/8$ (the beam is half-way to the top wall), in the case of $\sigma_x/w = 0.1$, $\sigma_y/h = 0.05$.

5 CONCLUSIONS

Non-linearities and corrections due to a finite transverse beam size in beam fields and BPM signals are calculated for a homogeneous vacuum chamber in the case when either the wavelength of interest is longer than a typical transverse dimension of the chamber and/or the beam is ultra relativistic. It is proved that *transverse beam-size corrections vanish in all orders for any azimuthally symmetric beam in an arbitrary chamber.* For other beam charge distributions they tend to be minimal when the distribution is more symmetric.

Explicit analytical expressions are derived for two particular cases, circular and rectangular chamber cross section, as well as for the particular beam charge distributions, double-Gaussian and uniform rectangular ones.

While it was not discussed here, the calculated corrections to beam fields can be directly applied for calculating beam coupling impedances produced by small discontinuities of the vacuum chamber using the method of Ref. [6].

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6 REFERENCES

[1] J.H. Cuperus, *Nucl. Instr. Meth.*, **145**, 219 (1977).

[2] R.H. Miller, et al., in *Proceed. of 12th Int. Conf. on High Energy Accel.* (Fermilab, 1983), p. 602.

[3] S.J. Russel and B.E. Carlsten, in *Proceed. of PAC* (New York, 1999), p. 477.

[4] R. Assmann, B. Dehning, and J. Matheson, in *Proceed. of EPAC* (Vienna, 2000), p. 1693; *ibid.*, p. 412; also in *AIP Conf. Proceed.*, **546**, 267 (2000).

[5] S.S. Kurennoy, Report SNS: 01-028, Los Alamos, 2001; to be published.

[6] S.S. Kurennoy, R.L. Gluckstern and G.V. Stupakov, *Phys. Rev. E*, **52**, 4354 (1995).