Quantisation of a relativistic particle in electromagnetic field

Dana Constantinescu[1], Radu Constantinescu[2], Marian Negrea[2]
University of Craiova
13 A.I. Cuza, 1100 Craiova, Romania

Abstract

We analyse the interaction of a high energy particle with an electromagnetic field. At the classical level the Lorentz-Dirac equation describes the relativistic motion of the charge. At higher frequencies quantum effects become significant. But the quantisation of the theory is difficult because the gauge invariance makes the Langrangean singular. We utilise the B.R.S.T. method of quantisation in the Batalin-Vilkoviski formulation and we substitute in the Lagrangean the original fields with the generalized ones who include also the ghost and the antifields. The resulting action satisfies the master equation and can be used like effective action of the theory.

1. INTRODUCTION

As well known an electric charge moving in an external field can generate a radiation field. This field interacts with the charge and this affects its motion. The result is the appearing of a new force, known as radiative reaction, force which produces the deceleration of the emitting charge.

Taking into account the radiative reaction, in the non- relativistic approximation, the equation of motion of pointlike particle of charge q and mass m has the form:

$$m\dot{\vec{v}} = \frac{1}{4\pi}\frac{2q^2}{3c^3}\ddot{\vec{v}} + \vec{F} \tag{1}$$

where $\vec{F}$ stands for all external forces applied to the charge.

The relativistic description of the same phenomenon can be done in a 4-dimensional Minkowski space. If we assume τ to be the proper time parameter of this space, we shall characterise the particle by the 4-component Lorentz vector $x^\alpha(\tau)$ and by the 4-vectors velocity $u^\alpha = dx^\alpha/d\tau \equiv \dot{u}^\alpha$ and acceleration $w^\alpha = \dot{u}^\alpha$. If we shall take into account the radiative reaction, the suitable relativistic equations will be the Lorentz-Dirac equations [1]:

$$m_0 w^\alpha = \frac{1}{4\pi}\frac{2q^2}{3c^3}\left(\frac{dw^\alpha}{d\tau} + \frac{u^\alpha w^2}{c^2}\right) + F^\alpha \tag{2}$$

Here m_0 represents the "renormalised" value of the rest mass of particle (finite and constant). It is easy to see that in the nonrelativistic limit $c \to \infty$ equation (2) coincides with (1).

In this paper will shall be interested in the description of the quantum effects of the motion of the relativistic particle in given external electromagnetic field. In this case F^α in (2) will represent the Lorentz force. We shall see that our model could be considered like a constraint system and so for the quantisation of his motion we can apply the general procedures utilise in QFT for the gauge theories.

A very important achievement in the quantisation of gauge theories in the path integral framework was obtained by Fadeev and Popov who have shown that a consistent quantum description of a system can be done if the gauge invariant action will be replaced in the path integral by an "effective action" which includes some additional variables, without a physical meaning. These variables have been denominated "ghost fields" and their role is to cancel the integration over the superfluos degree of freedom in the path integral. However, the Fadeev-Popov method doesn't work very well for a large class of gauge theories, as for example the models with an open gauge algebra. For all these models the only satisfactory method of quantisation appears to be the BRST method. This method is based on the so-called BRST symmetry, a global Grassmann odd symmetry for effective action. The BRST symmetry incorporates very interesting properties and plays for the effective gauge fixed action the same role as the gauge invariance for the gauge theory.

In the section 2 of this paper we shall develop the Lagrangean BRST formalism of Batalin and Vilkoviski [2,3,4]. In this formalism, on the space of the all variables of the model and of their associated "antifields", an "antibracket" can be defined. One obtain the so-called "antibracket-antifield formalism" and it will provide a powerful tool in the construction of a quantum theory for a physical system with a constrained dynamics. This formalism will be applied in the last section for a relativistic particle in a given external electromagnetic field. Since this example corresponds to an irreductible case, we shall restrict ourselfs to the presentation of the BRST quantisation technique only for this case. The radiative reaction will not be actually taked into account here. It will be analised in the BRST context elsewhere [5], like an interesting situation when the Lagrangean could depend on the field derivatives of order greater than 1: $L = L\left(q^i, \dot{q}^i, \ddot{q}^i, \dots\right)$.

1 Dept. of Mathematics
2 Dept. of Theoretical Physics

2. THE ANTIBRACKET-ANTIFIELD FORMALISM IN THE BRST QUANTISATION

Let us consider a gauge field theory for the fields q^i, i=1,...,N with the Grassmann parity $\varepsilon(q^i)=\varepsilon_i$. Let $S_0\left(q^i,\dot{q}^i\right)$ be the classical action describing the system. The equations of motion associated will have the form:

$$\frac{\delta S_0}{\delta q^i}=0 \quad (3)$$

The solutions q_0^i will define the stationary points and the set of this points forms the stationary surface Σ.

If the action is invariant under the gauge transformations

$$\delta q^i = R^i_\alpha \xi^\alpha(x) \qquad \alpha=1,...,N \quad (4)$$

the field equations will not be independent. One gets the local identities: $\frac{\delta S_0}{\delta q^i}R^i_\alpha=0$.

By integrating (4) on Σ one obtains the gauge orbits G. The gauge invariant observables will belong to the space $C^\infty(\Sigma/G)$. If the generators $R^i_\alpha(x)$ are all independent we have the case of an irreducible theory. By contrary, if there exist some non trivial λ^α such that $\lambda^\alpha R^i_\alpha \approx 0$ (equality on shell) we say that the theory is reducible.

The BRST symmetry is a global symmetry that presupposes the replacement of the original gauge invariance (4) by a new one, the BRST invariance. This global symmetry will be represented by an operator s which will act as a graded odd derivation on the original fields q^i and on some supplementary meaningless variables.

If we mark q^A the set of all variables, including q^i, we can define for each of these an "antifield" q^*_A, conjugated with q^A in relation to a special odd bracket, named "antibracket": $(q^A,q^*_B)=\delta^A_B$. For two arbitrary functionals the antibracket will be:

$$(F,G)=\frac{\delta F}{\delta q^A}\frac{\delta G}{\delta q^*_A}-\frac{\delta F}{\delta q^*_A}\frac{\delta G}{\delta q^A} \quad (5)$$

The most important properties of s are the following:

1) nilpotency: $s^2=0$
2) oddness: $\varepsilon(sF)=\varepsilon(F)+1$
3) the zeroth cohomological group contained all the gauge invariant observables: $H^0(s)=C^\infty(\Sigma/G)$
4) can be realised like a canonical transformation in antibracket: sF=(F,S)

S is named the canonical generator of s and its construction is equivalent with the construction of s. In the homological perturbation technique s starts with the so-called Koszul-Tate operator: $s=s_0+....$

In the same technique $S=\sum_n \overset{(n)}{S}$ and we must identify the first term with the classical action of the theory: $\overset{(0)}{S}=S_0(q^i)$. Passing now to the quantisation problem, we shall define a bosonic functional W, the quantum extension of S:

$W=S+\sum_{n=1}^{\infty}\hbar^n W_n$

We shall ask that $\frac{1}{2}(W,W)=i\hbar\Delta W$, where $\Delta=(-1)^{\varepsilon_A}\partial^A\partial_A$.

In the first order, from the last two equation we obtain the master equation: (S,S)=0 (6)

Solving this equation will be the key problem in the BRST quantisation of the theory. The antibracket - antifield quantisation achieves this in two steps:

1) One construct an "extended action" by a suitable choice of the spectrum of fields and antifields
2) by a gauge fixing procedure, putting all antifields equal to zero, one obtain the effective action of the theory S_{eff}. This one will be used in the path integral formalism for the construction of the functional integral

$$Z=\int dq^i dq^*_i \exp\left[\frac{i}{h}S_{eff}\right] \quad (7)$$

It is very interesting to remark that one fixes the gauge the antibracket structure disappears, without having a quantum analog. But this structure is very useful in the first step when one gets a remarkable symmetry between the fields q^A and antifields q^*_A.

Let us see the minimal spectrum needed to obtain a suitable effective action for an irreducible gauge theory.

-to each field equation (3) one defines the antifield q^*_i so that $s_0 q^*_i=-\frac{\delta S_0}{\delta q^i}$

-to each generator of the gauge transformations (4) we attach the antifield q^*_α defined by

$$s_0 q^*_\alpha = R^i_\alpha q^*_i$$

The symmetry field-antifield in relation to the antibracket will impose the introduction of the ghosts q^α, so that

$$\left(q^\alpha,q^*_\beta\right)=\delta^\alpha_\beta$$

With this spectrum, a proper solution of the master equation will have in our case the form :

$$S=S_0+q^*_i R^i_\alpha\cdot q^\alpha+...... \quad (8)$$

Given (8) one can always add pairs of variables, say λ^a,η^a without change the physical content of the generator S. We obtain a non-minimal solution of the master equation:

$$S_{\text{non min}}=S+\lambda^*_a\eta^a \quad (9)$$

The gauge fixed action will be obtained if we shall make the substitution

$$q^*_A=\frac{\delta\Psi(q)}{\delta q^A} \quad 10)$$

Where $\Psi(q)$ is a Grassmann odd function called the gauge fermion.

An important result concerns the independence of the functional integral (7) on the choice of the gauge fermion [4].

3. THE RELATIVISTE PARTICLE IN AN ELECTROMAGNETIC FIELD

Let us apply the general formalism developed in the last paragraph for the concrete case of a relativistic particle in a given electromagnetic field. In this section we shall use the units where $\hbar$=c=1 and the metric (+, -, -, -). We shall consider first a free relativistic particle. The dynamics of this system will be describes by the action:

$$S_0=-m\int_{\tau_1}^{\tau_2}d\tau\sqrt{-\dot{x}_\alpha\dot{x}^\alpha} \quad (11)$$

The Euler-Lagrange equations will be [6]:

$$L_\mu \equiv -m\frac{d}{d\tau}\frac{\dot{x}_\mu}{\sqrt{-\dot{x}_\alpha\,\dot{x}^\alpha}} = -m\frac{dp_\mu}{d\tau} = 0$$

where p_μ are the momenta conjugate to x^μ.

After a straightforward calculation one obtains that $\dot{x}^\mu L_\mu = 0$. Not all Euler-Lagrange equations are independent and the particle can be analysed like a constrained dynamical system. We are in the case of on irreducible system, the only one first class constraint that apear being:

$$G \equiv \frac{1}{2}(p^2 + m^2) \approx 0 \tag{12}$$

We can introduce this constraint in the action by a Lagrange multiplier e(t). We should then start from the action [4]:

$$S_0[x^\mu, p^\mu, e] = \int_{\tau_1}^{\tau_2} d\tau\left[p_\mu\,\dot{x}^\mu - \frac{e}{2}(p^2+m^2)\right] \tag{13}$$

Another posibility is to regard e(t) as a field variable with an conjugated momentum $\pi(t)$. We obtain in this case an extra first class constraint:

$$G_2 \equiv \pi(t) \approx 0 \tag{14}$$

If we use now λ^a, a=1,2 as Lagrange multipliers, an equivalent action with (13) will be:

$$S = \int d\tau\left(p_\mu\,\dot{x}^\mu + \pi\,\dot{e} - \frac{1}{2}\lambda^1(p^2+m^2) - \lambda^2\pi\right) \tag{15}$$

This action is invariant under the following set of irreducible gauge transformations:

$$\delta x^\mu = p^\mu\xi^1;\ \delta e = \xi^2;\ \delta\lambda^1 = \dot{\xi}^1;\ \delta\lambda^2 = \dot{\xi}^2$$

$$\delta p_\mu = 0;\ \delta\pi = 0$$

We can identify now the spectrum of the fields and antifields with this introduce in the previous section for the general case of an irreducible system: $q^i \equiv x^\mu$; $q^\alpha \equiv e$.

The set of nonminimal ghost contains the "fields" $\{\lambda^a; \eta^a\}$ a=1,2. The gauge fixed action will take the form:

$$S_{eff} = -m\int_{\tau_1}^{\tau_2} d\tau\left[p_\mu\,\dot{x}^\mu + \pi\,\dot{e} + \lambda_1^*\,\dot{\eta}^1 + \lambda_2^*\,\dot{\eta}^2 - \frac{e}{2}(p^2+m^2)\right.$$
$$\left. -\lambda_1^*\eta^2\right] \tag{16}$$

We shall pass now to the analyse of a relativistic particle in a given external electromagnetic field A^μ. The corresponding action of the system will be:

$$S = -m\int_{\tau_1}^{\tau_2} d\tau\sqrt{-\dot{x}_\alpha\,\dot{x}^\alpha} - q\int_{\tau_1}^{\tau_2}\dot{x}^\mu A_\mu d\tau \tag{17}$$

For the canonically conjugated momentum to x^μ will obtain the expression: $p_\mu = m\frac{\dot{x}_\mu}{\sqrt{-\dot{x}_\alpha\,\dot{x}^\alpha}} - qA_\mu$.

We can observe that again the canonical Hamiltonian give rise to a first class constraint:

$$\varphi_1 = (p_\mu + qA_\mu)^2 + m^2 \approx 0 \tag{3.14}$$

We could regard everything as the field gives a constant amount qA_μ to the particle momentum. The Euler-Lagrange equations are:

$$L_\mu = -m\frac{dp_\mu}{d\tau} + q\,\dot{x}^\nu F_{\nu\mu} = 0 \tag{3.15}$$

We shall start the B.R.S.T. quantisation of the system with an important remark: in the operator formalisme it is possible to make, when the electromagnetic field is present, the simple transition from $\hat{p}$ to $\hat{p}+q\hat{A}$ in the wave equation from the free particle. The justification becomes clear if we shall check the validity of the wave equation after this modification is done. Taking into account the gauge invariance of the electromagnetic field $A'_\mu = A_\mu + \partial_\mu f$ we shall easy see that the solution Ψ of the new wave equation can be written in the form:

$$\Psi = \Psi_0 e^{-iqf}$$

where Ψ_0 is the solution of the same equation for the free particle. The two solution contain the same physical informations. Therefore it is natural to generalise in the BRST approach too for the particle in an electromagnetic field the relations obtained for the free particle, using the same transition from p_μ to $p_\mu + qA_\mu$.

We shall rewrite the action (17) in the form:

$$S(x^\mu, e, \lambda^a) = \int d\tau\left(p_\mu\,\dot{x}^\mu + \pi\,\dot{e} - \lambda^1 G_1 - \lambda^2 G_2\right)$$

where

$G_1 \equiv (p_\mu + qA^\mu)^2 + m^2 \approx 0;\ G_2 = \pi \approx 0$.

We can verify that $(x^\mu, G_1) = 2(p^\mu + qA^\mu), (e, G_2) = 1$

The gauge fixed action will be in this case of the form:

$$S_{eff} = \int d\tau\left\{p_\mu\,\dot{x}^\mu + qA\,\dot{x}^\mu + \pi\,\dot{e} + \lambda_1^*\,\dot{\eta}^1 + \lambda_2^*\,\dot{\eta}^2 - \right.$$
$$\left. -e\left[(p_\mu + qA_\mu)^2 + m^2\right] - \lambda_1^*\eta^2\right\} \tag{18}$$

The action (18) is equivalent with the action (16) and it can be used in (7) for the quantum description of the relativistic particle in a given electromagnetic field.

4. REFERENCES

[1] Novozhilov, Yu. V and Yappa, Yu. A. - Electrodynamics Mir Publishers, Moscow, 1981

[2] Batalin, I.A. and Vilkoviski, G.A. - Phys Rev D28 (1983) 2567

[3] Henneax, M. - Preprint ULB 89-07

[4] Fisch, J.M.L. - Ph. D. shesis ULB TH21/90-01

[5] Constantinescu, R. - In preparation

[6] Sundermeyer, K. - Constrained Dynamics, Springer - Verlag 1982