Calculated Evolution of a Proton Bunch under RF-Noise

Sergei Ivanov
Institute for High Energy Physics
Protvino, Moscow region, 142284, Russia

Abstract

New aspects of longitudinal dilution of a bunch subjected to RF-noise are dealt with. Namely,

(a) Diffusion coefficient beyond stationary buckets is found that allows to treat the outer diffusion in the beam halo. So far, a trapped motion inside bucket, mostly in its center, has been only in question.

(b) An "absorbing wall" type of boundary condition is imposed at separatrix or at infinity (either may be adequate), the latter being the coordinate of a physical aperture. Throughout the region covered, the diffusion coefficient is essentially nonlinear, nonmonotonic and possibly diverges near separatrix. Earlier quantitative studies of bunch life-times employed its simplified model (linear-, quadratic- or power-functions against an action variable).

(c) Given an intricate diffusion coefficient and two types of boundary conditions, the initial-value boundary problem is treated numerically (for arbitrarily long bunches) via the Finite Element Technique, which allows to estimate bunch life-times. These emerge from the criteria of either bunch quality degradation or of its population loss. The latter depend on coordinate at which the boundary condition is imposed, whose effect is as well evaluated.

1 INTRODUCTION

Consider a bunched p-beam rotating in a storage ring at a constant energy. Let $\mathcal{J}$ be an action variable with $\mathcal{J}=\mathcal{J}_{sep}$ at separatrix. Use is made of the reduced variables

$$\mathcal{J}' = \mathcal{J}/\mathcal{J}_{sep}; \quad t' = t\,\Omega_s^2(0)\,\mathcal{P}_*^{(a,\varphi)}V_{ext}^{-2} \qquad (1)$$

with t the time, $\Omega_s(\mathcal{J}')$ the angular synchrotron frequency, V_{ext} the net amplitude of external RF-voltage, $\mathcal{P}_*^{(a,\varphi)}$ a representative value of spectral density $\mathcal{P}^{(a,\varphi)}(\Omega)$ of noise voltages. The latter are either $\Delta V^{(a)}=\Delta V_{ext}(t)$ for amplitude, or $\Delta V^{(\varphi)}=V_{ext}\Delta\varphi(t)$ for phase noises.

A bunch is given by its distribution $\langle F_0\rangle(\mathcal{J}',t')$ with $\langle\ldots\rangle$ the statistical average over a noise ensemble. Under an ergodicity assumption, it coincides with a particular bunch-evolution observation smoothed by a proper time-averaging. The '0'-subscript denotes the mathematical average over an angle ψ – canonical conjugate of $\mathcal{J}$.

For short-correlated stationary noises the evolution of $\langle F_0\rangle$ is known [1, 2, 3] to follow a diffusion equation:

$$\frac{\partial\langle F_0\rangle(\mathcal{J}',t')}{\partial t'} = \frac{\partial}{\partial\mathcal{J}'}\left(D(\mathcal{J}')\frac{\partial\langle F_0\rangle(\mathcal{J}',t')}{\partial\mathcal{J}'}\right). \qquad (2)$$

2 DIFFUSION COEFFICIENT

2.1 *Inside the Bucket* ($\mathcal{J}'\leq 1$)

Diffusion coefficient $D(\mathcal{J}')$ is given parametrically via a pair $D(x)$, $\mathcal{J}'(x)$; $0\leq x\leq 1$. It is a weighted sum over the noise harmonics

$$D^{(a,\varphi)}(x) = \frac{\pi^2}{128}\sum_{m=-\infty}^{\infty}\mathcal{W}^{(a,\varphi)}(x)\left(\frac{\mathcal{P}(m\Omega_s(x))}{\mathcal{P}_*}\right)^{(a,\varphi)}; \qquad (3)$$

$$\mathcal{J}'(x) = \mathrm{E}(x) - (1-x^2)\,\mathrm{K}(x); \quad \frac{\Omega_s(x)}{\Omega_s(0)} = \frac{\pi}{2\mathrm{K}(x)}. \qquad (4)$$

Both the parity and strength of the bunch multipole excitations depend on weight-factors

$$\mathcal{W}_m^{(a,\varphi)}(x) = (1\pm(-1)^m)^2\left(\frac{\pi m}{2\mathrm{K}(x)}\right)^4 \qquad (5)$$
$$\times\begin{cases}\sinh^{-2}\left(\dfrac{\pi m}{2\mathrm{K}(x)}\mathrm{K}(\sqrt{1-x^2})\right). & (a)\\ \cosh^{-2} & (\varphi)\end{cases}$$

Here cosh, sinh are the hyperbolic functions; $\mathrm{K}(x)$, $\mathrm{E}(x)$ are the complete elliptic integrals of modulus x.

In a White-Noise approximation $\mathcal{P}^{(a,\varphi)}(\Omega)\simeq\mathcal{P}^{(a,\varphi)}$ is factored out. The series left in Eq.3 is summable:

$$D^{(a,\varphi)}(x) \simeq \frac{\pi^2}{128}\mathcal{A}^{(a,\varphi)}(x), \qquad \mathcal{P}_*^{(a,\varphi)} = \mathcal{P}^{(a,\varphi)}; \qquad (6)$$

$$\mathcal{A}^{(a,\varphi)}(x) = \frac{16}{\pi^2}\mathrm{K}(x)\Bigg(\frac{15\pm 1}{30}\Big(\mathrm{E}(x)-(1-x^2)\mathrm{K}(x)\Big) \mp\frac{4}{15}x^2(1-x^2)\Big(2\mathrm{E}(x)-\mathrm{K}(x)\Big)\Bigg). \qquad (7)$$

These Eqs. were first got in refs.[1, 2, 3]. For their alternative derivation based on the approach of [4] refer to [5].

Fig.1 plots inner functions $2\mathcal{W}_m(\mathcal{J}')$, $\mathcal{A}(\mathcal{J}')$. The latter diverges logarithmically near separatrix.

Consider the center of bucket ($\mathcal{J}'\lesssim 0.5$). Expand $\mathcal{J}'(x)$, $D(x)$ in Taylor series in x^2. Invert these sums to get a quadratic approximation:

$$D^{(a,\varphi)}(\mathcal{J}') \simeq \begin{cases}\frac{1}{4}\mathcal{J}'^2, & (a)\\ \frac{\pi}{16}\left(\mathcal{J}'-\frac{3}{\pi}\mathcal{J}'^2\right), & (\varphi)\end{cases} \qquad (8)$$

given $\mathcal{P}_*^{(a)}=\mathcal{P}^{(a)}(2\Omega_s(0))$; $\mathcal{P}_*^{(\varphi)}=\mathcal{P}^{(\varphi)}(\Omega_s(0))$.

Eq.2 can now be reduced to a closed (ordinary first-order differential) equation in terms of the $\langle F_0\rangle$-averaged

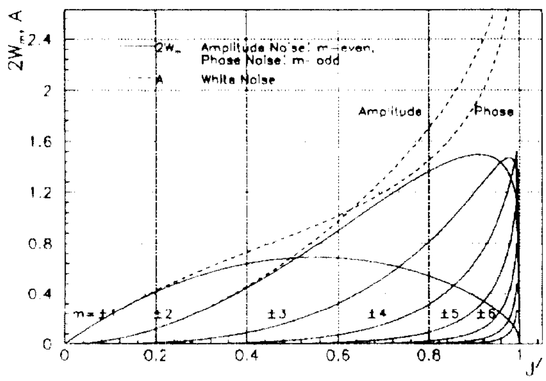

Figure 1: Weight and amplitude functions (inner).

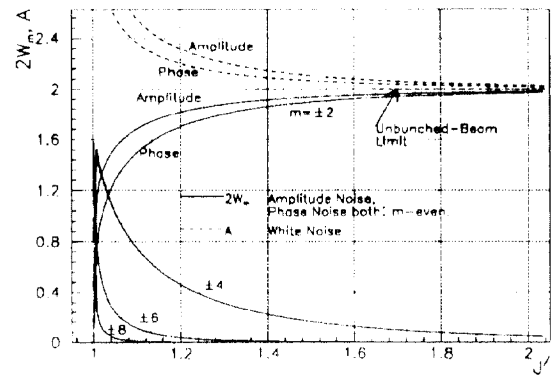

Figure 2: Weight and amplitude functions (outer).

longitudinal emittance $\overline{\mathcal{J}'}$. Its solutions for $t' \geq 0$ are

$$\overline{\mathcal{J}'}^{(a)}(t') = \overline{\mathcal{J}'}(0)\exp(+\frac{1}{2}t'); \tag{9}$$

$$\begin{aligned}\overline{\mathcal{J}'}^{(\varphi)}(t') &= \frac{\pi}{6} + \left(\overline{\mathcal{J}'}(0) - \frac{\pi}{6}\right)\exp(-\frac{3}{8}t') \qquad (10)\\ &\rightarrow \overline{\mathcal{J}'}(0) + \frac{\pi}{16}t' \quad \text{as} \quad \overline{\mathcal{J}'}(0), t' \rightarrow 0.\end{aligned}$$

These Eqs. describe the onset of a short-bunch quality degradation (no loss of beam population yet). The last line is a common result for linear oscillations. For more involved application of this, "moment-of-distribution" techniques to Eqs.2,8 refer to [3]. The same ref. offers fundamental solutions of Eq.2 with $D \sim \mathcal{J}'$ or $\mathcal{J}'^2$. Formally, [3] puts zero boundary condition at ∞, and extends thus far the small-amplitude Eqs.8. The approach [2] is to erect an "absorbing wall" at separatrix, and treat a problem resulting for $D \sim \mathcal{J}'^n$, $n < 2$ by Fourier-Bessel expansion.

2.2 *Beyond the Buckets ($\mathcal{J}' \geq 1$)*

Let M identical bunches be placed in all h stationary buckets available. Diffusing particles would eventually abandon the buckets to continue an orbital motion in their outer vicinity. On averaging over any RF-imposed period, distribution of particles in the beam halo can formally be interpreted as the outer continuation of $\langle F_0\rangle(\mathcal{J}', t')$. (Naturally, in case of $M > 1$ it can well be made of particles which have never belonged to the bunch at issue.)

Evolution of $\langle F_0\rangle(\mathcal{J}' > 1, t')$ is shown in [5] to be governed by the same diffusion Eq.2, but with a new $D(\mathcal{J}' > 1)$. To save the paper length, we list only the formal substitutions which extend Eqs.3-7 into the $(M = h)$-beam halo region; $1 \leq x \leq \infty$:

$$\begin{aligned}\mathrm{K}(x) &\rightarrow x^{-1}\mathrm{K}\left(x^{-1}\right);\\ \mathrm{K}\left(\sqrt{1-x^2}\right) &\rightarrow x^{-1}\,\mathrm{K}\left(\sqrt{1-x^{-2}}\right); \qquad (11)\\ \mathrm{E}(x) &\rightarrow x\mathrm{E}\left(x^{-1}\right) + (1-x^2)x^{-1}\mathrm{K}\left(x^{-1}\right);\\ (1 \pm (-1)^m)^2 &\rightarrow (1 + (-1)^m)^2.\end{aligned}$$

Fig.2 plots the thus got outer functions $2\mathcal{W}(\mathcal{J}')$, $\mathcal{A}(\mathcal{J}')$. Visually, curves $2\mathcal{W}^{(a,\varphi)}$ merge pair-wise as $|m| \geq 4$.

The far off-set particles ignore V_{ext} and move as in an unbunched beam. It can be shown that the noise heating-up of an unbunched beam by $\Delta V^{(a,\varphi)}$ is indeed described by $(\mathcal{J}' \rightarrow \infty)$-asymptotes of $\mathcal{W}(\mathcal{J}')$; $\mathcal{A}(\mathcal{J}')$.

3 BOUNDARY CONDITIONS

Statement of the initial-value diffusion problem is completed by imposition of boundary conditions:

– Loss of particles at some distance $\mathcal{J}' = \mathcal{J}'_{\max}$:

$$\langle F_0\rangle(\mathcal{J}'_{\max}, t') = 0. \tag{12}$$

– Continuity of both $\langle F_0\rangle(\mathcal{J}', t')$ and of diffusion flux:

$$Q(\mathcal{J}', t') = -D(\mathcal{J}')\,\frac{\partial\langle F_0\rangle(\mathcal{J}', t')}{\partial\mathcal{J}'} \tag{13}$$

at separatrix $\mathcal{J}' = 1$ whenever $\mathcal{J}'_{\max} > 1$.

– No source of particles in the bunch center:

$$Q(\mathcal{J}' = 0, t') = 0. \tag{14}$$

If $M \ll h$ (like in Sp$\bar{\text{p}}$S) a reasonable suggestion is $\mathcal{J}'_{\max} = 1$ [2]. Particles in the halo rarely reappear near the filled buckets. On crossing separatrix they are effectively lost.

However, if $M = h$ a more adequate assumption is $\mathcal{J}'_{\max} \rightarrow \infty$. The particles are lost forever at a practically infinitely distant physical aperture.

Both these problems, i.e. for $\mathcal{J}'_{\max} = 1; \infty$, are treated numerically. For a $(M \leq h)$-beam these would yield lower and upper bounds for population-loss life-times.

4 NUMERICAL SOLUTION

4.1 *Method*

A finite-difference $\mathcal{J}'$-discretization does not suit. Troubles arise in computation of $Q(\mathcal{J}', t')$ at points $\mathcal{J}' = 0; 1$, where $D = 0; \infty$. Instead, use is made of the Finite Element Method with piece-wise linear basic interpolation functions [6]. The weak form of the Galerkin's weighted residual techniques readily accepts natural boundary conditions imposed in terms of the flux values, Eq.13,14.

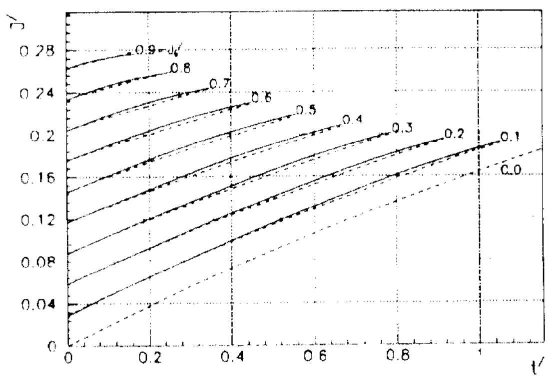

Figure 3: Emittance growth (phase noise).

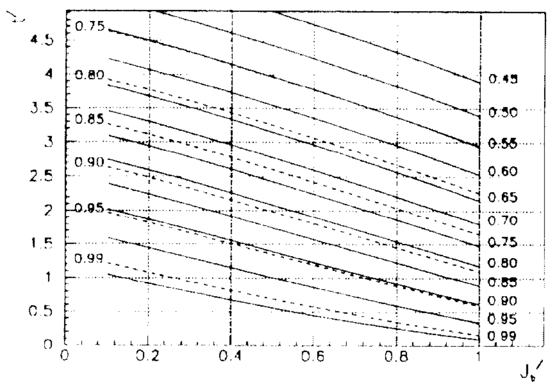

Figure 5: Population-loss life-times (phase noise).

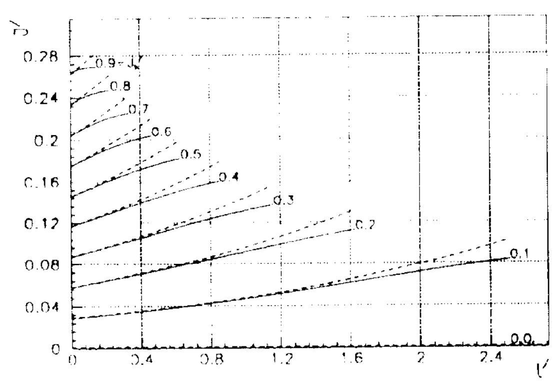

Figure 4: Emittance growth (amplitude noise).

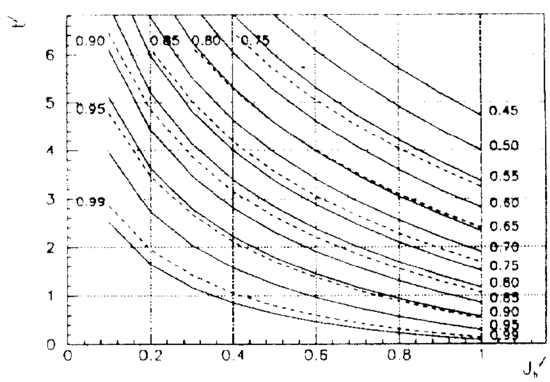

Figure 6: Population-loss life-times (amplitude noise).

Integration in t' is performed via the finite-difference Crank-Nicolson's scheme [6]. This 2-layer procedure is unconditionally stable. For sufficiently small step $\Delta t' \sim \Delta \mathcal{J}'^2/D$ (which is estimated in practice via solution of a generalized eigenvalue problem for 3-diagonal matrices) a numerical solution is, as well, free of unwanted oscillations.

To interpret the solutions as these for $\mathcal{J}'_{\max} \to \infty$, the t'-integration proceeds until the distribution's tail touches some distant, though finite, aperture at $\mathcal{J}'_{\max} > 1$. If necessary, this boundary is shifted further, etc.

4.2 Results

Presented here are the results under a White-Noise assumption, for (a)- and (φ)-noises individually.

Figs.3,4 show the computed evolutions of the bunch emittances $\overline{\mathcal{J}'}$ (solid lines). They are compared to the analytical solutions of Eqs.9,10 (dashed lines). The parameter plotted is the initial bunch-sise at base $\mathcal{J}'_b(0)$. Injected distribution $\langle F_0\rangle(\mathcal{J}', 0)$ is combined of 2 parabolas conjugated at $\frac{1}{2}\mathcal{J}'_b(0)$. Each curve is cut off at time $t'_{0.99}$ until which a more than 0.99-fraction of the initial population is left captured in the bucket, given $\mathcal{J}'_{\max}= 1$. Notice an accurate fit by Eq.10-curve for the (φ)-noise, which is accounted for by 2-term Taylor expansion in Eq.8. These plots estimate the life-times related to onset beam-quality degradation.

The further bunch dilution is covered by Figs.5,6 which show the computed population-loss life-times vs. $\mathcal{J}'_b(0)$ for the same $\langle F_0\rangle(\mathcal{J}', 0)$ injected. Parameter is a fraction of population left inside the bucket. Solid lines (parameter value at the right), and dashed lines (parameter at the left) stand for the boundary problems with $\mathcal{J}'_{\max}= 1$ or ∞, respectively. The more time elapses, the more significant is the effect of boundary conditions (i.e., finally, of the beam orbital structure) on the life-times.

The same approach is applied to narrow-band noises which offer more intricate functions $D(\mathcal{J}')$. The feed-backs introduce another complications: $D(\mathcal{J}'; \langle F_0\rangle)$. These are as well treatable numerically.

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