A Taylor-Expanded Generating Function for Particle Motion in Arbitrary Magnetic Fields

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Abstract

Analytical expressions for a Taylor-expanded generating function describing the particle motion in an arbitrary magnetic field are presented. The expansion parameters are the transverse momentum coordinates and the inverse of the bending radius at a reference point. The expansion converges very rapidly for large bending radii. It can be applied to any kind of magnetic field such as fringe fields of multipole magnets or undulator fields. Based on an analytical representation of the magnetic vector potential this generating function is usefull for a fast and fully canonical tracking routine.

1 INTRODUCTION

In [1] a second order generating function for particles in magnetic fields has been derived and applied as a fast tracking routine across undulator fields. In that paper a vector potential that vanishes at the beginning and the end of the transformation interval is assumed. This generating function is applied to the field of a planar undulator, where the transversal vector potential is zero at the plane of the poles, limiting the step size to multiples of half a period length. Nagaoka et al. [2] derived a generating function for nonplanar insertion devices that allows arbitrary step sizes by taking the transversal vector potential into account. This is of special interest for long period and strong field devices in low energy storage rings [3].

In this article we present the second order expansion of the generating function for an arbitrary magnetic field without any restrictions concerning the vector potential.

The calculations in this article have been performed with the algebraic code REDUCE [4].

2 THE ALGEBRAIC MAP

The particle motion in the magnetic field is described by an algebraic map over a finite integration interval z. First we derive a Taylor-expanded map in a fixed Cartesian coordinate system. The expansion parameters are the two transverse angle variables (x_i', y_i') at the starting point of the transformation interval and a third variable x_3, the inverse of the bending radius of the particle orbit taken at a reference point. This set of three expansion variables is rather unusual, but as shown in [1] they could form a fast converging series which does, at low order, already include effects of higer order multipoles.

The general form of a Taylor expanded map along the z axis with respect to the variables x_i' , y_i' and x_3 has the form:

$$x = x_i + z\cdot x_i' + \sum_{k,l,m} a_{klm}\cdot x_i'^k \cdot y_i'^l \cdot x_3^m \quad (1)$$

$$x' = x_i' + \sum_{k,l,m} a'_{klm}\cdot x_i'^k \cdot y_i'^l \cdot x_3^m \quad (2)$$

$$x'' = \sum_{k,l,m} a''_{klm}\cdot x_i'^k \cdot y_i'^l \cdot x_3^m \quad (3)$$

$$y = y_i + z\cdot y_i' + \sum_{k,l,m} b_{klm}\cdot x_i'^k \cdot y_i'^l \cdot x_3^m \quad (4)$$

$$y' = y_i' + \sum_{k,l,m} b'_{klm}\cdot x_i'^k \cdot y_i'^l \cdot x_3^m \quad (5)$$

$$y'' = \sum_{k,l,m} b''_{klm}\cdot x_i'^k \cdot y_i'^l \cdot x_3^m \quad (6)$$

Derivatives with respect to the longitudinal coordinate z are indicated by a prime. The inital transverse particle coordinates x , x' , y and y' are indicated by an index i. We exclude any coupling of the two transverse planes where the magnetic field is switched off ($x_3 = 0$); in this case we obtain just the drift transformation. In second order the indices k, l, m are restricted to $1 \leq k+l+m \leq 2$. The coefficents a_{klm} and b_{klm} need to be calculated. They depend on the position coordinates at the starting point, on the longitudinal position z and on the geometric shape $\vec{R}$ of the vector potential.

The shape function $\vec{R}$, the expansion variable x_3, the magnetic vector potential $\vec{A}$ and the magnetic rigidity of the particle $(B\rho)$ are connected by the relation:

$$\vec{R}\cdot x_3 = \vec{A}/(B\rho);$$

$\vec{R}\cdot x_3$ is dimensionless, and the expansion converges rapidly if this expression is small. The parameter x_3 can be interpreted as the inverse of the bending radius at some reference point (x_0, y_0, z_0), scaling with the strength of the vector potential, whereas the function $\vec{R}$ is independent of the strength and describes the geometric shape of the magnetic vector potential.

The Taylor series expansion of the three components of the shape function around the starting position coordinates x_i, y_i has the form ($u = x$,y or z):

$$R^u = \sum_{n_1,n_2} R^u_{n_1 n_2 0}\cdot (x-x_i)^{n_1}\cdot (y-y_i)^{n_2}/(n_1!\cdot n_2!)$$

The z dependent coefficients

$$R^u_{n_1n_2n_3} = \partial^{n_1+n_2+n_3}R^u/\partial x^{n_1}y^{n_2}z^{n_3}$$

are evaluated at the position $(x = x_i, y = y_i, z)$. Later we will need the functions $R^{ui}_{n_1n_2n_3}$ which are the derivatives of the shape function $R^u_{n_1n_2n_3}$ at the point $z = 0$. The transverse coordinates x and y are defined by Eq.1 and Eq.4. A multipole expansion of the shape function with respect to x_i and y_i is not necessary. We keep the functional dependence of the shape function on the initial particle position. This improves the overall convergence of the series. In this way an expansion with only three parameters is possible.

The magnetic field follows from the shape function by:

$$\vec{B}/B\rho = x_3 \cdot rot(\vec{R})$$

Inserting the Taylor series expansion of the shape function $\vec{R}$ we get an expansion for the magnetic field.

Now we insert this field expansion as well as the coordinate expansion given by Eq.2,3,5,6 into the equations of motion for a particle in a Cartesian coordinate system [5]:

$$\begin{aligned} x'' &= \sqrt{} \cdot [y'B_z - (1+x'^2)B_y + x'y'B_x]/(B\rho) \\ y'' &= -\sqrt{} \cdot [x'B_z - (1+y'^2)B_x + x'y'B_y]/(B\rho) \\ \sqrt{} &= \sqrt{1+x'^2+y'^2}. \end{aligned}$$

Comparing the coefficients of products of equal order formed by x'_i, y'_i, x_3 yields a set of equations from which the coefficients a_{klm} and b_{klm} can be determined in a recursive way. In second order the second derivatives of the coefficients have the values:

$$\begin{aligned} a''_{001} &= -(R^x_{001} - R^z_{100}) \\ b''_{001} &= -(R^y_{001} - R^z_{010}) \\ a''_{002} &= -(a_{001} \cdot R^x_{101} - a_{001} \cdot R^z_{200} \\ &\quad + b'_{001} \cdot R^x_{010} - b'_{001} \cdot R^y_{100} \\ &\quad + b_{001} \cdot R^x_{011} - b_{001} \cdot R^z_{110}) \\ b''_{002} &= a_{001} \cdot R^z_{110} - a_{001} \cdot R^y_{101} \\ &\quad + R^x_{010} \cdot a'_{001} - R^y_{100} \cdot a'_{001} \\ &\quad - b_{001} \cdot R^y_{011} + b_{001} \cdot R^z_{020} \\ a''_{011} &= -(z \cdot R^x_{011} - z \cdot R^z_{110} + R^x_{010} - R^y_{100}) \\ b''_{011} &= -(z \cdot (R^y_{011} - R^z_{020})) \\ a''_{101} &= -(z \cdot (R^x_{101} - R^z_{200})) \\ b''_{101} &= z \cdot R^z_{110} - z \cdot R^y_{101} + R^x_{010} - R^y_{100} \end{aligned}$$

The coefficients a_{klm} and b_{klm} are obtained by integrating twice along the z-axis over the step length z. The transformation:

$$(x_i, x'_i, y_i, y'_i) \implies (x_f, x'_f, y_f, y'_f)$$

is now well defined and these four equations form a second order map between the inital and final particle coordinates.

In the presentation of the coefficients we see that the order $k + l + m$ of the coefficients on left hand side is larger than the order on the right hand side. This is a general property of the expansion and allows the recursive calculation of the coefficients. The reason of this property can be seen in the equation of motion, where x'' and y'' are at least proportional to the expansion variable x_3.

Suppose we would expand the particle coordinates x and y with respect to the more common set of expansion variables x_i, x'_i, y_i, y'_i

$$\begin{aligned} x &= x_i + z \cdot x'_i + \sum_{k,l,m,n} a^*_{klmn} \cdot x^k_i \cdot x'^l_i \cdot y^m_i \cdot y'^n_i \\ y &= y_i + z \cdot y'_i + \sum_{k,l,m,n} b^*_{klmn} \cdot x^k_i \cdot x'^l_i \cdot y^m_i \cdot y'^n_i \end{aligned}$$

and further expand the equations of motion to the same basis. Then the equations for the determination of the coefficients a^*_{klmn} and b^*_{klmn} are complicated coupled differential equations which in general can not be solved analytically. This stresses the importance of the three parameter expansion we use in this article that results in a recursion procedure that can be solved much more easily. Also, the three variable expansion requires fewer coefficents to be defined.

3 THE GENERATING FUNCTION

To derive the generating function we change the cartesian coordinates (x, x', y, y') into canonical variables (qx, px, qy, py) as follows:

$$\begin{aligned} qx &= x \\ px &= A^x/(B\rho) + x'/\sqrt{1+x'^2+y'^2} \\ qy &= y \\ py &= A^y/(B\rho) + y'/\sqrt{1+x'^2+y'^2}. \end{aligned}$$

The two transverse expansion variables are now the canonical momenta px and py. In the further calculation we express the vector potential $\vec{A}$ via its shape function $\vec{R}$ as defined above.

At the starting point of the integration interval the vector $\vec{R}$ is given by the initial position coordinates $\vec{R}^i = \vec{R}(x_i, y_i, z = 0)$. At the endpoint of the transformation interval we use the Taylor series expansion of the shape function which has already been defined.

We obtain now a transformation:

$$(qx_i, px_i, qy_i, py_i) \implies (qx_f, px_f, qy_f, py_f)$$

which will be expanded with respect to px_i, py_i, x_3. An inversion of the series and an expansion with respect to px_f, py_f, x_3 yields the implicit transformations:

$$(qx_i, px_f, qy_i, py_f) \implies (qx_f, px_i, qy_f, py_i)$$

with expressions of the type:

$$\begin{aligned} qx_f &= \sum_{k,l} Q_{x,kl} \cdot px^k_f \cdot py^l_f \\ px_i &= \sum_{k,l} P_{x,kl} \cdot px^k_f \cdot py^l_f \end{aligned}$$

and similarly for qy_f and py_i.

In second order the expansion coefficients $Q_{x,kl}$, $Q_{y,kl}$, $P_{x,kl}$ and $P_{y,kl}$ have the form:

$$
\begin{aligned}
Q_{x,00} &= x_3^2 \cdot (z^2 \cdot R^x_{010} \cdot R^y_{000} + z^2 \cdot R^x_{000} \cdot R^x_{100} \\
&\quad - z \cdot R^x_{010} \cdot \tilde{b}_{001} - z \cdot R^x_{100} \cdot \tilde{a}_{001} - a'_{001} \cdot \tilde{a}_{101} \\
&\quad - R^x_{000} \cdot \tilde{a}_{101} - b'_{001} \cdot \tilde{a}_{011} - R^y_{000} \cdot \tilde{a}_{011} + \tilde{a}_{002}) \\
&\quad + x_3 \cdot (-(z \cdot R^x_{000}) + \tilde{a}_{001}) + qx_i \\
Q_{x,10} &= -(x_3 \cdot (z^2 \cdot R^x_{100} - \tilde{a}_{101}) - z) \\
Q_{x,01} &= -(x_3 \cdot (z^2 \cdot R^x_{010} - \tilde{a}_{011})) \\
Q_{y,00} &= x_3^2 \cdot (z^2 \cdot R^y_{100} \cdot R^x_{000} + z^2 \cdot R^y_{000} \cdot R^y_{010} \\
&\quad - z \cdot R^y_{100} \cdot \tilde{a}_{001} - z \cdot R^y_{010} \cdot \tilde{b}_{001} - a'_{001} \cdot \tilde{b}_{101} \\
&\quad - R^x_{000} \cdot \tilde{b}_{101} - b'_{001} \cdot \tilde{b}_{011} - R^y_{000} \cdot \tilde{b}_{011} + \tilde{b}_{002}) \\
&\quad + x_3 \cdot (-(z \cdot R^y_{000}) + \tilde{b}_{001}) + qy_i \\
Q_{y,10} &= -(x_3 \cdot (z^2 \cdot R^y_{100} - \tilde{b}_{101})) \\
Q_{y,01} &= -(x_3 \cdot (z^2 \cdot R^y_{010} - \tilde{b}_{011}) - z) \\
P_{x,00} &= x_3^2 \cdot (z \cdot R^x_{010} \cdot R^y_{000} + z \cdot R^x_{000} \cdot R^x_{100} \\
&\quad - R^x_{010} \cdot \tilde{b}_{001} + a'_{101} \cdot a'_{001} + a'_{101} \cdot R^x_{000} \\
&\quad + a'_{011} \cdot b'_{001} + a'_{011} \cdot R^y_{000} - a'_{002} - R^x_{100} \cdot \tilde{a}_{001}) \\
&\quad + x_3 \cdot (-a'_{001} - R^x_{000} + R^{xi}_{000}) \\
P_{x,10} &= -(x_3 \cdot (z \cdot R^x_{100} + a'_{101}) - 1) \\
P_{x,01} &= -(x_3 \cdot (z \cdot R^x_{010} + a'_{011})) \\
P_{y,00} &= x_3^2 \cdot (z \cdot R^y_{100} \cdot R^x_{000} + z \cdot R^y_{000} \cdot R^y_{010} \\
&\quad - R^y_{100} \cdot \tilde{a}_{001} + a'_{001} \cdot b'_{101} + R^x_{000} \cdot b'_{101} \\
&\quad + b'_{001} \cdot b'_{011} + R^y_{000} \cdot b'_{011} - b'_{002} - R^y_{010} \cdot \tilde{b}_{001}) \\
&\quad + x_3 \cdot (-b'_{001} - R^y_{000} + R^{yi}_{000}) \\
P_{y,10} &= -(x_3 \cdot (z \cdot R^y_{100} + b'_{101})) \\
P_{y,01} &= -(x_3 \cdot (z \cdot R^y_{010} + b'_{011}) - 1)
\end{aligned}
$$

with the abbreviation $\tilde{a}_{klm} = a_{klm} - z \cdot a'_{klm}$ and similarly for $\tilde{b}_{klm}$. At this point we could recombine the products $x_3 R^u_{n_1 n_2 n_3}$ to terms $A^u_{n_1 n_2 n_3}/B\rho$ and become independent of the definition of x_3.

From the generating function $F(qx_i, qy_i, px_f, py_f)$ excact canonical transformations are obtained:

$$
\begin{aligned}
qx_f &\equiv \partial F/\partial px_f & \qquad px_i &\equiv \partial F/\partial qx_i \\
qy_f &\equiv \partial F/\partial py_f & \qquad py_i &\equiv \partial F/\partial qy_i.
\end{aligned}
$$

Using the coefficients $Q_{x,kl}$, $Q_{y,kl}$, $P_{x,kl}$ and $P_{y,kl}$ we construct the following generating function:

$$
\begin{aligned}
F &= F_{00} + F_{10} \cdot px_f + F_{01} \cdot py_f + \\
&\quad F_{20} \cdot px_f^2 + F_{11} \cdot px_f \cdot py_f + F_{02} \cdot py_f^2 \\
F_{00} &= \int P_{x,00} dqx_i + s(qy_i)
\end{aligned}
$$

$$
\begin{aligned}
F_{10} &= Q_{x,00} & \qquad F_{01} &= Q_{y,00} \\
F_{20} &= Q_{x,10} & \qquad F_{11} &= Q_{x,01} \\
F_{02} &= Q_{y,01}
\end{aligned}
$$

$s(qy_i)$ is an integration constant that is choosen such that $\partial F_{00}/\partial qy_i = P_{y,00}$. The way of determining the parameters F_{ij} is not unique. But the result is always the same, if the function $\vec{A}$ is consistent with the Maxwell equations. This can be checked by the following relations:

$$
\begin{aligned}
\partial P_{x,00}/\partial qy_i &= \partial P_{y,00}/\partial qx_i \\
\partial Q_{x,00}/\partial qx_i &= P_{x,10} + O(2) \\
\partial Q_{x,00}/\partial qy_i &= P_{y,10} + O(2) \\
\partial Q_{y,00}/\partial qx_i &= P_{x,01} + O(2) \\
\partial Q_{y,00}/\partial qy_i &= P_{y,01} + O(2) \\
Q_{x,01} &= Q_{y,10}
\end{aligned}
$$

O(2) are expressions of the order 2 and more in the expansion parameter x_3.

The partial derivatives of this generating function yield a canonical transformation, which, in second order, is identical to the transformation derived above; but additionally higher order terms appear.

For a given magnetic field the vector potential $\vec{A}$ is well defined apart from a rotation free vector $\vec{P}$. We checked the results above for a transformation $\vec{A} \Longrightarrow \vec{A} + \vec{P}$ with $rot(\vec{P}) = 0$. From the canonical transformation above we can derive a Taylor expanded transformation of the cartesian coordinates with the expansion parameters x'_i, y'_i and x_3. It turns out that the transformation of the cartesian coordinates is affected by the transformation of the vector potential only in third and higher orders in the parameters x'_i, y'_i and x_3.

4 CONCLUSION

We derived the second order generating function for the transformation of a charged particle through a magnetic field. In principle the method can be expanded to higher orders. But the formulas are getting quite long even in third order. We demonstrated that the choice of the expansion parameters is essential. The equations of motion can be integrated in a recursive way when we use the two transverse momenta and the inverse of a bending radius as expansion parameters.

5 REFERENCES

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