## DENSITY WAVES DRIVEN BY A HIGH HARMONIC CAVITY* <br> Joseph M. Kats <br> Brookhaven National Laboratory <br> Bldg. 911B <br> Upton, NY 11973, USA

## ABSTRACT

A high harmonic cavity (HHC) with a certain phase modulation frequency can increase the longitudinal emittance and smooth the density distribution within a particle bunch. Such a cavity helps to prevent particle losses during passage through the transition energy. The speed of dilution and the quality of density redistribution depend on the phase modulation program. In this report, we show how to choose such a program which will create traveling waves which push the front of high density from the center of the bunch to its boundary. Supporting results from machine and computer experiments are presented.

## 1. Basic Equations and Parameters

The synchrotron motion of particles within a stationary bucket in absence of the HHC is governed by the equations:
$-2 \frac{\partial \mathrm{H}_{\mathrm{o}}}{\partial \phi_{\mathrm{O}}}-2 \delta \mathrm{E}_{\mathrm{O}}=\operatorname{Sin} \phi_{\mathrm{O}}: \quad\left|\delta \mathrm{E}_{\mathrm{O}}\right| \leq 1$,
$\frac{d \mathrm{H}_{0}}{\partial \delta \mathrm{E}_{0}}-\dot{\phi}_{0}--2 \delta \mathrm{E}_{0}, \quad\left|\phi_{0}\right| \leq \pi$
with corresponding Hamiltonian
$H_{0}=-\delta E_{0}^{2}-\operatorname{Sin}^{2} \frac{\phi_{0}}{2}$.

In the presence of HHC synchrotron equations are:
$-2 \frac{\partial \mathrm{H}}{\partial \phi}-2 \delta \mathrm{E}-\sin \phi+\varepsilon \sin (\mathrm{N} \phi-\psi), \delta \mathrm{E}^{2} \leq 1+\frac{\epsilon}{\mathrm{N}}$,
$\partial \mathrm{H}$
$\overline{\partial \delta E}=\phi-2 \delta \mathrm{E}, \quad|\phi| \leq \pi$
with Hamiltonian
$H--\delta E^{2}-\sin ^{2} \frac{\phi}{2}-\frac{\varepsilon}{N} \sin ^{2} \frac{\mathrm{~N} \phi-\psi}{2}$.
Both systems above are written in dimensionless form, where dimensionless time $T-\Omega_{0} t$ is measured in units of synchrotron period $T_{o}-2 \pi / \Omega_{0}$, $\quad-d / d r$,
$\delta \mathrm{E}-\left(\mathrm{E}-\mathrm{E}_{\mathrm{s}}\right) / \mathrm{E}_{\mathrm{b}}, \mathrm{E}_{\mathrm{b}}^{2}-2 q \mathrm{~V} / \pi a-\left(2 \Omega_{\mathrm{o}} / a \omega\right)^{2}$,
$a-h|\eta| / \beta_{s_{\mathrm{E}}}^{\mathrm{S}_{\mathrm{s}},} \eta^{\prime}=1 / \gamma_{\mathrm{t}}^{2}-1 / \gamma_{\mathrm{s}}^{2}, \varepsilon=\mathrm{V}_{\mathrm{H}} / \mathrm{V}$.
Subscripts $s, t$ refer to synchronous and transition energy, subscript $b$ refers to bucket, $E_{b}$ is bucket half-height, $q$ is particle charge, $V$ is (total) peak voltage of main if system, driven with the frequency $f$
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which is synchronized with the particle's revolution frequency $\omega$ by $h \omega-2 \pi f, h$ is the machine harmonic number ( $h-12$ for the AGS), $V_{H}$ is the peak voltage of HHC driven by the frequency $\mathfrak{f}_{\mathrm{H}}, \mathrm{f}_{\mathrm{H}} / \mathrm{f}=\mathrm{N}$ is HHC harmonic number ( $N=21+1 / 3$ for the $A G S$ ).

Phase modulation produced by $H H C$ is determined by the modulation program:

$$
\begin{equation*}
\psi-\psi(\tau)=\alpha \operatorname{Cos} \gamma \tau \tag{1.7}
\end{equation*}
$$

with amplitude $\alpha$ [radians] and frequency $\gamma$ [mults plier of synchrotron frequency $\Omega_{0}$ ].

## 2. The Resonance Equation

The exact solution of unperturbed equations (1.1), (1.2) for any, say i-th particle, can be writ ten by means of elliptic functions. ${ }^{1}$ To avold complex technicalities, we will introduce an approximate solution which is more convenient for the subsequent analysis:

$$
\begin{equation*}
\phi_{o i}(\tau)=r_{i} \cos \left(k_{i} \tau+\beta_{i}\right) . \tag{2.1}
\end{equation*}
$$

where $r_{1}$ is amplitude, $\operatorname{Sin}^{2}\left(r_{1} / 2\right)-\delta \mathrm{E}_{\mathrm{oi}}^{2}(0)+$
$\sin ^{2}\left(\phi_{\mathrm{oi}}(0) / 2\right), \beta_{i}$ is initial polar angle, $\beta_{i}-$ $\operatorname{arctg}\left[\delta \mathrm{E}_{\mathrm{Oi}}(0) / \phi_{O i}(0)\right]$, and $\mathrm{k}_{1}=k\left(r_{1}\right)$ is synchioutron frequency, which can be approximated as ${ }^{2}$

$$
\begin{equation*}
k-k(r)-\sqrt{1-r^{2} / \pi^{2}} \tag{2.2}
\end{equation*}
$$

Let us assume that two particles start motion from the same position ( $\delta E_{1}, \phi_{1}$ ). Let the first particle move under unperturbed conditions (1.1), (1.2), while the second particle moves under the influence of HHC according to (1.4), (1.5). Let
$\Delta E=\delta E-\delta E_{0}, \Delta \phi-\phi-\phi_{0}$
Subtracting (1.1), (1.2) from (1.4), (1.5) and expanding $\delta E, \phi$ around $\delta E_{0}, \phi_{0}$ to the first order of $\Delta E, \Delta \phi$, we will get
-•
$\Delta \phi+\left[\operatorname{Cos} \phi_{0}+\varepsilon N \operatorname{Cos}\left(N \phi_{0}-\psi\right)\right] \Delta \phi--\varepsilon \operatorname{Sin}\left(N \phi_{0}-\psi\right)$,

$$
\begin{equation*}
\Delta \phi(0)=\Delta \phi(0)=0 . \tag{2.4}
\end{equation*}
$$

Thus, for every particle in the initial given bunch, there is a corresponding system (2.4), (2.5). Each system is specified by two parameters: inftial ampli. tude $r_{i}$ and polar angle $\beta_{i}$.

If we select all the particles having the same amplitude, say $r_{i}-r$, then we can average (2.4), (2.5) over such an ensemble, where particles differ only by theit angles $\beta_{i}$. Denoting that type of averaging by angular brackets and assuming initial distribution as stationary we find:

$$
\begin{equation*}
\left\langle\operatorname{Cos}\left(N \phi_{0}-\psi\right)\right\rangle-J_{0}(N r) \operatorname{Cos} \psi, \quad\left\langle\operatorname{Cos} \phi_{0}\right\rangle=J_{0}(r), \tag{2.6}
\end{equation*}
$$

$$
\begin{equation*}
\left\langle\operatorname{Sir}\left(N \phi_{0}-\psi\right)\right\rangle=-J_{0}(N r) \operatorname{Sin} \psi . \tag{2.7}
\end{equation*}
$$

where $J_{0}$ is the Bessel function of zero order
After the averaging (let us drop an angular bzacket and remember it by having Bessel functions as a reminder) Eq. (2.4) becomes
-•
$\Delta \phi+\left[J_{0}(r)+\varepsilon N J_{0}(N r) \operatorname{Cos} \psi\right] \Delta \phi=\varepsilon J_{0}(N r) \operatorname{Sin} \psi$.
Because Fourler series for $\operatorname{Sin} \psi$, Cos $\psi$ are slow con verging, we expand them within a segment $0 \leq \phi \leq 4$ by the least squares method into the sums
$\cos \phi=c_{0}(\alpha)+c(\tau)=$
$c_{0}(\alpha)+c_{2}(\alpha) \operatorname{Cos} 2 \gamma \tau+c_{4}(\alpha) \operatorname{Cos} 4 \gamma \tau+c_{6}(\alpha) \operatorname{Cos} 6 \gamma \tau$,
$\operatorname{Sin} \psi-S(\tau)=$
$s_{1}(\alpha) \operatorname{Cos} \gamma \tau+s_{3}(\alpha) \operatorname{Cos} 3 \gamma \tau+s_{5}(\alpha) \operatorname{Sin} 5 \gamma \tau$,
where $c_{i}=c_{i}(\alpha), s_{j}-s_{j}(\alpha)$ are known (from least squares) polynomials of 6 th (for $c_{i}$ ) and 5 -th (for sj) orders, $i=0,2,4,6 ; j=1,3,5$. Then the Eq. (2.8) becomes Hill's-type
-•
$\Delta \phi+\left(\omega^{2}+h N\left(c_{2} c_{2}+c_{4} c_{4}+c_{6} c_{6}\right)\right) \Delta \phi-$
$h\left(s_{1} C_{1}+s_{3} C_{3}+s_{5} C_{5}\right)$,
where $C_{m}-\operatorname{Cosin} \gamma \tau$,

$$
\begin{equation*}
\mathrm{h}=\varepsilon J_{0}(\mathrm{Nr}), w^{2}=J_{0}(r)+\mathrm{hNc}_{0}(\alpha) \tag{2,12}
\end{equation*}
$$

The solution of (2.11), satisfying (2.5) is a sum of homogeneous and nonhomogeneous parts. Neglecting harmonics of higher than 6 -th order, one can write a solution as
$\Delta \phi(T)=A\left(e^{\mu t}+\sigma e^{-\mu t}\right)\left(a_{o}+C_{a}+S_{b}\right)+C_{p}$,
where
$C_{a}-a_{2} C_{2}+a_{4} C_{4}+a_{6} C_{6}, C_{p}-p_{1} C_{1}+p_{3} C_{3}+p_{5} C_{5}$,
$s_{b}=b_{2} s_{2}+b_{4} s_{4}+b_{6} s_{6}, \quad s_{m}=\sin m \gamma \tau$.

The resonance strength $\mu=\mu(\gamma)$ is an eigenvalue of the homogeneous equation, corresponding to (2.11); constants $A$ and $o$ are determined from initial conditions (2.5). A real-valued $\mu$ exists for any $\gamma$ and increases wioti $\gamma$ up to $\gamma-3$ after which the increase is negligible.

## 3. Dilution Criterion

In phase space, the area within closed particle's trajectory is the particle's enittance. It is connected monotonically with the particle's Hamiltonian. The bigger the Hamiltonian absolute value, the bigger the emittance. So the Hamiltonian's derivative can be taken as a criterion of dilution.

By differentiating (1.6) with use of (1.4), (1.5) we have dilution criterion (local in time) for any given particle:

$$
\begin{equation*}
\frac{\mathrm{dH}}{\mathrm{~d} \tau}-\frac{\varepsilon}{2 \mathrm{~N}} \sin (\mathrm{~N} \phi-\psi) \psi<0 . \tag{3.1}
\end{equation*}
$$

If the modulation $\psi$ is just a simple harmonic as fn (1.7), then the maximal speed of dilution can be estimated as
$\left|\frac{d H}{d \tau}\right| \leq \frac{\varepsilon}{2 N}|\dot{\psi}|=\frac{\alpha \gamma \varepsilon}{2 N}$.
By decomposing $\phi-\phi_{0}+\Delta \phi$ and by averaging over the ensemble of particles having the same initial amplitude $r$, we can represent (3.1) as
$\underset{\mathrm{d} T}{\stackrel{\mathrm{dH}}{\longrightarrow}}-\frac{\varepsilon}{2 \mathrm{~N}} \mathrm{~J}_{0}(\mathrm{Nr}) \operatorname{Sin}(N \Delta \phi-\psi) \phi$,
or, for harmonic modulation (1.7), as
$\xrightarrow[d \tau]{\stackrel{d H}{>}}=-\frac{\alpha \gamma \varepsilon}{2 N} J_{0}(N r) \operatorname{Sin}(N \Delta \phi-\phi) \operatorname{Sin} \gamma \tau$
or
$\stackrel{\mathrm{dH}}{\mathrm{d} \tau} \underset{4 \mathrm{~N}}{\mathrm{~L}}-\frac{\alpha \gamma \varepsilon}{\left.\mathrm{J}_{0}(\mathrm{Nr}) \cos (\mathrm{u}-\gamma \tau)-\cos (\mathrm{u}+\gamma \tau)\right),}$
where
$u=u(\tau)-N \Delta \phi(\tau)-\alpha \operatorname{Cos} \gamma \tau$.

In (3.5) the expression in square brackets is the difference of two non-linear phase waves. Each front moves with velocity $\dot{i}-\gamma$ and $\dot{s}-\gamma$ respectively.

The averaging over ensemble $<>$ is not good enough to make the dilution criterion other than instantenous, which is of little value for an oscil. lating process. In order to make a working tool, we have to average <dH/dt> over a characteristic period of oscillations. In other words, we have to average over $T_{\gamma}-2 \pi / \gamma$. We will denote that type of averaging by a bar.

By expanding cosines in (3.5) into the power series up to the second order and double averaging over $\mathrm{T}_{\boldsymbol{\gamma}}$, one can write
$\stackrel{\overline{d H}}{\stackrel{\mathrm{dr}}{\longleftrightarrow}} \sim-\frac{\alpha \gamma^{2} \varepsilon}{2} J_{0}(\mathrm{Nr}) \tau \Delta \phi$.
By extracting the main term from ap one gets
$\overline{\overline{d^{2}}} \overline{d \tau}>-J_{o}(N r) c(\alpha) \tau e^{\mu t}$,
where $c(\alpha)$ is a function independent of $T$ yet dependent on all the coefficients in (2.13). The sign of $c(\alpha)$ is like the sign of Bessel function $J_{1}(\alpha): J_{1}(\alpha)$ $>0$ if $0<\alpha<3.8, J_{1}(\alpha)<0$ if $3.83<\alpha<7$.

The relationship (3.8) does not hold for instantaneous or long term behavior. It holds during times comparable with the synchrotron period. After that, the particles which were described by $(3,8)$ can change their amplitude $r$ so much that $r$ can not be considered as independent of $\tau$ and all applications of Section 2 should be repeated for a new r
starting with a new $\tau=0$

> 4. Discussion of the Results

We have seen that the dilution criterion depends on the product $J_{o}(N r) J_{1}(\alpha)$, where $r$ is the particle amplitude and $\alpha$ is the phase modulation amplitude. If $a$ is a constant over a dilution cycle then the sign of the product is determined by $J_{0}(\mathrm{Nr})$. Shown fn Fig. 1, the whole range of amplitudes, $r$ is broken by $J_{0}(N r)$ into segments in such a way that $J_{0}$ has opposite signs for any two adjacent scgments. Then the phase space, occupied by a bunch, will be broken into rings, If there is a dilution in one ring $\left(J_{0} J_{1}>\right.$ 0) then there is a contraction in the next one. The strength of dilution (contraction) is proportional to the peak amplitude of $J_{0}(\mathrm{Nr})$ over the ring area. The strength decreases from bunch center to the boundary.


Fig. 1: An oscillating function $J_{o}(\mathrm{Nr})$ creates dilu. tirg and contracting rings.

There are three mechanisms contributing to bunch dilution. The first is "microstatistic". Nonlinearity of the motion creates noise-like effects which increases bunch emittance slowly. The second is resonance. Either systematic ${ }^{3}$ or parametric (like one was described here) resonances change particles, amplitudes quickly. The third is "macrostatistic". The rings of intermittent dilution create two flows of particles, moving towards the bunch center and back with the velocities; approaching zero at the boundaries of rings. Due to decreasing dilution strength from center-to-boundary, the flow from the center to the boundary will always prevail over opposite flow.

However, near the bunch boundary the difference of strengths (max $\|_{0}(\mathrm{Nr}) \mid$ ) of two adjacent rings is smaller than near the bunch center. This allows us to conserve bunch emittance while dilution is in effect. All we need to do is calculate $J_{0}\left(\mathrm{Nr}_{\mathrm{b}}\right)$ for the initial bunch with $\mathrm{r}_{\mathrm{b}}$ as bunch boundary amplitude. Next we choose $\alpha$ to satisfy
$J_{0}\left(\operatorname{Nr}_{b}\right) J_{1}(\alpha)<0$.
This will make the boundary ring contracting and bunch emittance conserving. Figures 2 and 3 show machine ${ }^{4}$ and computer experimenes.

Fig. 2: Left is "conserving" emittance, right is not.



Fig. 3: Left is "conserving" emittance, right is not
The bottom curve in a Fig. 2 is the line Inten sity of an initial bunch. The subsequent 5 curves ascending, are taken during 100 ms of dilution. The dashed lines in Fig. 3 are just the initial line density shown for comparison. The solid lines are calculated during the dilution. Within the bunches of particles, shown as dots, we can display a circle repre. senting the initial emittance.

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