DESCRIPTION OF NONLINEAR BEAM DYNAMICS IN THE CERN LARGE HADRON COLLIDER BY USING NORMAL FORM ALGORITHMS

A. Bazzani
Dipartimento di Fisica della Università di Padova
Istituto Nazionale di Fisica Nucleare - Sezione di Padova

B. Leemann and W. Scandale
CERN/SPS Geneva

G. Servizi and G. Turchetti
Dipartimento di Fisica della Università di Bologna
Istituto Nazionale di Fisica Nucleare - Sezione di Bologna

Introduction

The study of nonlinear effects due to the multipole errors of the superconducting magnets, is a crucial issue for the design of the hadron accelerators of the next generation. The standard method based on tracking program is usually limited by the computing time avaliable. In general the numerical simulation is performed over a few hundred turns, thus the dynamical parameters related to the nonlinearities such as the tune shift with the amplitude or the smear of the Courant-Snyder invariant, can be computed with a moderate accuracy.

A new approach based on Birkhoff normal forms was recently proposed in order to speed-up the simulation of the nonlinear motion both in LHC[1] and in SSC[2]. In this note we shortly summarize the main properties of the normal forms and we describe the code which allows to compute them by using algorithmic manipulations of polynomials. Some numerical results are presented, relative to the two dimensional motion in a quite realistic model of the LHC. A comparison with the results of a different tracking code is discussed. The future perspectives of the normal forms approach are analized.

Normal forms approach

The betatronic motion of a test particle in a given section of a particle accelerator can be described by a symplectic map $\mathcal{M}$, obtained by composing the transfer maps of all magnetic elements. It is well known that each transfer map can be approximated by a polynomial, whose order N is not less than the higest multipole error taken into account. If we compose sequentially the transfer maps truncating them each time at order N, we obtain the order N polynomial expansion $\mathcal{M}_N$ of the superperiod map. In the Floquet coordinates $\mathcal{M}_N$ reads:

$$\mathcal{M}_N(\vec{x}) = \mathcal{R}(2\pi\vec{\nu})\Big(\vec{x} + \sum_{n\geq 2}^{N} M_n(\vec{x})\Big) \qquad \vec{x} \in R^4 \qquad (1-1)$$

where $\mathcal{R}(2\pi\vec{\nu})$ is the direct product of two rotations in the phase planes $(x,p_x),(z,p_z)$, $\vec{\nu}$ are the linear tunes of the machine and M_n are homogeneus polynomial maps of order n. The map $\mathcal{M}_N$ is a truncation of a symplectic map, and cannot be used for tracking.

If the linear tunes $\vec{\nu}$ are non resonant, the Birkhoff theorem allows to construct a polynomial transformation Φ of order N which brings the map $\mathcal{M}_N$ into a symplectic map $\mathcal{N}$ according to the equation:

$$\Phi^{-1} \circ \mathcal{M}_N \circ \Phi = \mathcal{N} + E_N \quad E_N = O(\|\vec{X}\|^{N+1}) \qquad (1-2)$$

The non linear map $\mathcal{N}$ is a Birkhoff normal form and can be written as:

$$\vec{X}' = \mathcal{R}\big(\vec{\Omega}(R_X^2, R_Z^2)\big)\vec{X} \qquad (1-3)$$

where $R_X^2 = X^2 + P_X^2 \quad R_Z^2 = Z^2 + P_Z^2$. The extension of the Birkhoff theorem beyond the two dimensional case, was recently achieved using the generating function[3].

The tuneshift with amplitude and the smear are computed as follows: we choose an initial condition in the Floquet space and we compute the new coordinates (X,P_X,Z,P_Z) and the invariants R_X, R_Z by using the transformation Φ. Then by a simple polynomial evaluation we obtain:

$$\delta\vec{\nu} = \frac{\vec{\Omega}(R_X^2,R_Z^2)}{2\pi} - \vec{\nu} \qquad (1-4)$$

The smear depends on the section of the machine we are considering (just as the β function). A convenient definition of it is:

$$\begin{cases} \sigma_x = \dfrac{(< r_x^4 > - < r_x^2 >^2)^{\frac{1}{2}}}{< r_x^2 >} \\ \sigma_z = \dfrac{(< r_z^4 > - < r_z^2 >^2)^{\frac{1}{2}}}{< r_z^2 >} \end{cases} \qquad (1-5)$$

Namely is the mean square value of $r_x^2 = x^2 + p_x^2$ and $r_z^2 = z^2 + p_z^2$.

The limits of the method at the present stage are the following:

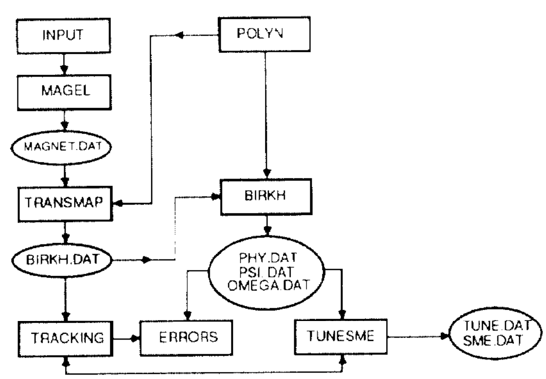
i) the dynamic aperture defined as the boundary of the attractor basin of infinity, can only be roughly estimated from the behaviour of the series defining the transformation Φ;

ii) in the vicinity of a resonance of order k, the Birkhoff can be used up to the order k only; higher order approximation are possible by using the resonant normal form whose development is in progress;

iii) the quality of the approximation can be evaluated only numerically; an analytical approach, not yet achieved, will allow rigorous estimates of the lifetime of the circulating intensity of particles.

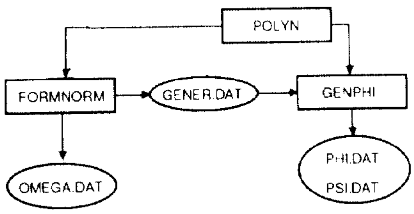
Description of the code

The main advantage of our method is the possibility of using an algorithmic approach to polynomial algebra. We have developed programs[4] which perfom very efficiently all the algebraic operations on polynomials (multiplication, power, composition, inversion...) including the computation of any polynomial function (exponential, logarithms..). The flow of the program is described in the following flowchart:



The INPUT routine for the magnetic data is not in standard form[5]; the MAGEL section computes the transfer maps of a single magnets in the kick approximation and write them on a file MAGNET.DAT. The program TRANSMAP concatenates the magnetic elements of the lattice and writes on a file BIRKH.DAT the coefficients of the superperiod map (the Floquet coordinates are used in a complex form: $w_x = x + ip_x, w_z = z + ip_z$). The main program BIRKH[6] computes the normal form transformation functions $\Phi, \Psi = \Phi^{-1}$ and the phase advances $\vec{\Omega}$ and write them on the files PHI.DAT,PSI.DAT and OMEGA.DAT. The program ERRORS computes the truncation error E_N given by eq.(1-3) allowing to check that its elements of order less or equal to N are zero within the machine accuracy, and gives other internal consistency checks for the Birkhoff series. The program TUNESME computes the tuneshift and the smear according to our definition eq.(1-4),(1-5).

The internal structure of BIRKH is described in the second flowchart:



The file GENER.DAT contains the coefficients of the generating function of the transformation Φ which is computed together with its inverse Ψ by GENPHI that solves a system of implicit equations. All the variables used in the program are in double precision (8 bytes for each real variable). The present release of the code is limited to four-dimensional maps and can be used up to order 10. Reaching higher orders depends on better managment of the memory and on vectorizing the basic algoritms. Up to order 8 the program has been interactively used on a 8600 VAX at CERN. A one dimensional version of the program was already written [7] and the current calculations were carried out at order 15 interactively on a 8600 VAX but higher order could be easily reached[1]. The extension of the program to a six-dimensional case does not present any further difficulty.

LHC model

We have considered a model for the LHC with a four-fold periodicity of the lattice. The superperiod consists of an arc, a low β insertion with $\beta^* = 0.5\,\mathrm{m}$, another arc and a high β insertion with $\beta^* = 4.0\,\mathrm{m}$. Each arc contains $24 + \frac{1}{2}$ standard FODO cells. Magnetic imperfections are considered to be present only in the superconducting dipoles. Their amplitude is identical in each dipole (systematic effect) and is represented by one kick approximation located in the middle of the magnet. The multipoles expressions of the imperfections are the following:[8]

$$\frac{B^{II}}{B_0\rho} = -2.4 \times 10^{-3}\,\mathrm{m}^{-3};\ \frac{B^{III}}{B_0\rho} = .01\,\mathrm{m}^{-4}$$
$$\frac{B^{IV}}{B_0\rho} = 44.8\,\mathrm{m}^{-5};\ \frac{B^{VI}}{B_0\rho} = -3.6 \times 10^{6}\,\mathrm{m}^{-7};$$
$$\frac{B^{VIII}}{B_0\rho} = 1.34 \times 10^{12}\,\mathrm{m}^{-9}$$

The chromatic corrections are computed by means of an auxiliary program. Our reference section is at the beginning of a defocusing quadrupole in a FODO. The basic parameters of our LHC model are given in [9]; the tunes are respectively $Q_x = 69.20, Q_z = 69.12$.

The numerical results obtained by a direct iteration for few superperiods of $\mathcal{M}_N$ and the normal form $\Phi \circ \mathcal{N} \circ \Phi^{-1}$ with $N = 8$, are compared with those of an independent tracking code (FASTRAC)[10]; we report the relative errors for two sets of initial values (in $\mathrm{cm}^{1/2}$) in Floquet space:

In. val.	Iter.	Trunc. map	Norm. form
$x = z = .03$	1	5.3×10^{-7}	8.4×10^{-6}
$p_x = p_z = 0$	10	8.7×10^{-6}	3.8×10^{-6}
$x = z = .06$	1	4.1×10^{-5}	2.7×10^{-3}
$p_x = p_z = 0$	10	1.5×10^{-4}	8×10^{-4}

We remark that the error of the truncated map grows linearly as we iterate the map both in the angular and radial coordinates because of the non symplectic character of $\mathcal{M}_N$. Conversely the error in the orbit computed with the normal form has three sources:

i) the order N at which we have truncated the superperiod transfer map $\mathcal{M}$ which gives an error proportional to r^{N+1} with $r = \max(r_x, r_z)$;

ii) the transformation function Φ;

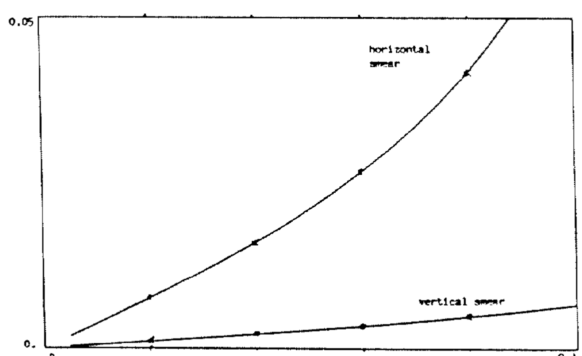
iii) the phase advance $\vec{\Omega}$. The first error is rougly evaluated for few turns of the machine. The second and the third error depends on the topology of the orbits and on the effects of the non-linear resonances. Anyway due to the properties of the normal form, the error in the radial coordinates depends very weakly on the number of iterations (see [11,12]).

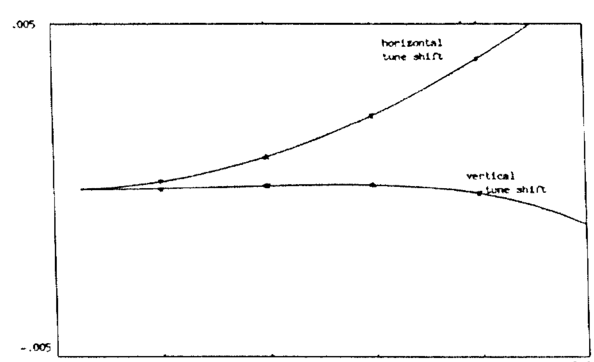
In table 2 we quote the l^2-norm of $\Phi^{(N)}$: i.e. the contribution at order N of the transformation Φ, which us proposed to be used as a measure of the non linearity:

Order	l^2norm
2	87.89×10^{-2}
3	25.57
4	39.71
5	43.80×10^{2}
6	39.68×10^{2}
7	74.52×10^{4}
8	62.29×10^{4}

The ratios $\sqrt{\frac{\|\Phi_n\|}{\|\Phi_{n+2}\|}}$ of the l_2-norms of the perturbative orders of the transformation Φ give a pseudo radius of convergence $r_c \approx 0.08\mathrm{cm}^{1/2}$, which is related to the closest low order resonance. In our case this is of the same order of magnitude of the dynamical aperture determined with FASTRAC (for 400 turns): $.12\,\mathrm{cm}^{-\frac{1}{2}}$ in the Floquet space.

In the figures we report, for the above model of LHC, the smear and the tuneshift computed with the normal form of order 8 for the following initial conditions in the Floquet space: $p_x = p_z = 0,\ 0 < x < .1,\ z = x/10$ for the horizontal plane and $p_x = p_z = 0,\ 0 < z < .1,\ x = z/10$ for the vertical plane.





In the same figure we quote the corresponding values calculated by FASTRAC with 400 turns.

We observe that there is an excellent agreement between the tracking and the normal form for both the smear and the tune shift. In any case by comparing the normal form at orders 6, 7, 8 we conclude that the errors at order 8 are much smaller than the errors affecting the tracking results.

Perspective

The method proposed is very efficient in computing the tune shift with the amplitude and the smear of a superconducting hadron collider as the LHC, although its application is presently limited to the non resonant working points and to systematic field imperfections. With some effort this method can be extended to include the evaluation of the resonances effects, the stochastic variation of the magnetic imperfections, and the beam lifetime.

The theoretical framework for analyzing the almost (or exact) resonant case already exists and numerical implementation will be easy. The analysis of a stochastic machine could be made by using anyway the normal forms, however we believe that a deeper theoretical understanding of the stochastically perturbed maps, by extending some of the ideas developed for the stochastic differential equations, need to be reached in order to develop efficient and reliable computational tools. For the beam lifetime some rough lower bound estimates can already be given at the present stage since the error for one turn of the normal forms dynamics behaves as:

$$\mathcal{E}_N = A_N \left(\frac{\epsilon}{\epsilon_c}\right)^{\frac{N+1}{2}}$$

where ϵ_c is about $5 \cdot 10^{-3}$ for our model. The beam lifetime is thus of order $1/\mathcal{E}_N$. Rigorous estimates could be given by generalizing the a priori estimates on the error which lead to the Nekhoroshev theorem[10] for hamiltonian flows and the exponential estimate for the long time diffusion.

The hardest problem still to be solved is the evaluation of the dynamical aperture: the only way to do that should be the analysis of unstable fixed points and manifolds. In the twodimensional case the normal forms proove to be succesfull in computing the unstable manifold. There is a hope that such results could be extended to high dimensional case allowing dynamical aperture to be obtained.

REFERENCES

[1] A.Bazzani, P.Mazzanti, G.Servizi, G.Turchetti: *LHC note 66* - CERN SPS 88 2 (1988)

[2] E.Forrest: *Hamiltonian-free perturbation theory* - SSC - 111 preprint (1987)

[3] A.Bazzani: *Normal forms for symplectic maps in R^{2n}* - Celestial Mechanics to be published (1987)

[4] A.Bazzani, G.Servizi, G.Turchetti: *POLYN: a program for algorithmic manipulation of polynomials* - in preparation (1988)

[5] F. C. Iselin, J. Niederer: *The MAD Program* - CERN-LEP-TH-87-33 (1987)

[6] A.Bazzani, G.Servizi, G.Turchetti: *BIRKH: a program for computing the Birkhoff normal forms for symplectic maps of R^{2n}* - in preparation (1988)

[7] G.Servizi, G.Turchetti: - Computer Physic Comunication 32 201 (1984)

[8] W. Scandale: - CERN SPS-87-28 (AMS) (1987)

[9] W. Scandale: *The lattice of the CERN-LHC for high luminosity p-p collisions* - LHC Note 68 (1988)

[10] B. Leemann, W. Scandale: *Performance limitations of the low-β insertion of the LHC* - EPAC Confernce (1988)

[11] G. Turchetti: ''*Perturbative methods for hamiltonian maps*'' - Proceedings of the Summer School of Medellin 1986 Ed by W.A. Saenz, World Scientific (1988)

[12] N.N. Nekhoroshev: - Russ. Math. Surveys 32*N*6 1 (1977)