Beam Optics of a Dipole Fringe Field

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Abstract

Solutions to the equations of motion of a charged particle through the fringing field of a bending magnet are represented by first, second, and third order transfer matrices. The case of the extended fringe field of a dipole with a straight inclined boundary is considered.

1 Introduction

The optical effects of the field boundaries of a bending magnet have been described to the second order for the sharp cutoff approximation [1] and to the first order for the extended fringe fields [2,8]. In the third order, the sharp cutoff approximation produces infinities in the matrix elements [7,9].

This paper outlines the procedure to obtain transfer matrices up to the third order for the extended fringe field of a dipole. We consider the case of a uniform field with straight inclined boundaries. The desired matrix elements are the coefficients in the Taylor expansion,

$$X_a^{final} = \sum_b R_{ab} X_b + \sum_b \sum_c T_{abc} X_b X_c + \sum_b \sum_c \sum_d U_{abcd} X_b X_c X_d + \cdots, \tag{1}$$

where $\mathbf{X}$ is the usual TRANSPORT [3] 6-vector, $\mathbf{X} = (x, x', y, y', l, \delta)$.

The matrices R, T, U in (1) depend on the pole face rotation angle ψ and some form factors. These form factors are line integrals of complicated functions of the field strength and its derivatives. To make the integrals tractable, one can expand them in a power series of $\epsilon = d/\rho$, where ρ is the inside bending radius and d is a measure of the fringe field extent.

2 Transfer Matrix Calculation

2.1 Formulation

We consider the entrance of the bending magnet shown in Fig. 1.

The effect of the fringe field of an inclined boundary is mathematically equivalent to the thin lens placed next to the magnet face normal to the design trajectory [1,8]. The transfer matrix for such a lens is given by a product of three transformations,

$$\mathcal{M}^{0\mapsto f} = \mathcal{M}^{2\mapsto f} \mathcal{M}^{1\mapsto 2} \mathcal{M}^{0\mapsto 1}, \tag{2}$$

where

1. $\mathcal{M}^{0\mapsto 1}$ is a transformation from the reference plane to the beginning of the fringe region through the pure drift field;
2. $\mathcal{M}^{1\mapsto 2}$ is the transformation through the fringe region;
3. $\mathcal{M}^{2\mapsto f}$ is the transformation from the end of the fringe region back to the reference plane through the pure bend field.

Pure drift and bend maps are well known to the third order [3]. We will calculate the non-trivial map $\mathcal{M}^{1\mapsto 2}$ here. The complete map $\mathcal{M}^{0\mapsto f}$ is found in [9].

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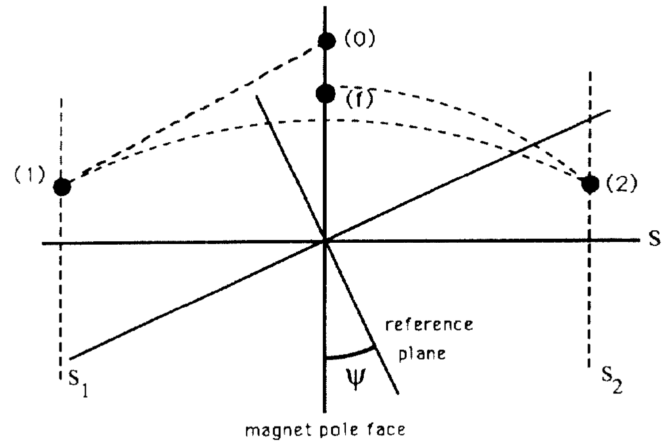

Figure 1: Midplane geometry. Reference plane is normal to the design trajectory.

The approach is as follows. First, we calculate the transfer map matrix elements for the canonical set of the phase space variables following Lie algebraic approach of [5]. Then, we transform to the TRANSPORT variables obtaining the desired elements in (1). What remains is performing matrix multiplication (2) and expanding the form factors in a power series of ϵ.

2.2 Fringe Region Map

The geometry of the problem is shown in Fig. 2. We would like to relate the coordinates at $s = s_2$ with those at $s = s_1$. We assume that the magnetic field goes smoothly from zero at s_1 to the constant value B_0 at s_2.

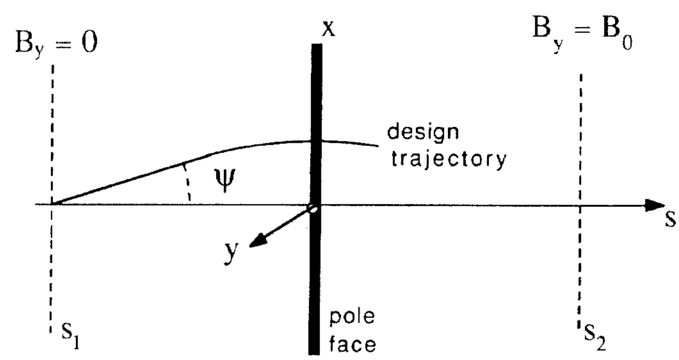

Figure 2: Fringe region at the entrance of a dipole.

The map $\mathcal{M}^{1\mapsto 2}$ has a unique Lie Algebraic factorization [4,5],

$$\mathcal{M} = \ldots exp(:f_4:)exp(:f_3:)exp(:f_2:), \tag{3}$$

where each $f_n(\mathbf{Z})$ is a homogeneous polynomial of degree n in $\mathbf{Z}$, $\mathbf{Z} = \mathbf{z} - \mathbf{z}^d$, $\mathbf{z} = (q_1, p_1, q_2, p_2, q_3, p_3)$ is a generalized coordinate-momentum

6-vector, and $\mathbf{z}^d$ is the design trajectory. Lie Transformation $exp(:f:)$ is defined in terms of the Poisson bracket operator, $[f,\ ]$,

$$exp(:f:)g = \sum_{n=0}^{\infty} \frac{:f:^n g}{n!} = g + [f,g] + \frac{1}{2}[f,[f,g]] + \cdots$$

Polynomials f_2, f_3, f_4 satisfy the equations of motion given in [7]. They are determined by the system's Hamiltonian and the first order matrix M given by $exp(:f_2:)\mathbf{Z} = M\mathbf{Z}$.

2.2.1 Fringe Region Hamiltonian

Hamiltonian for the fringe region with s as an independent variable can be written as follows,

$$K = -[(1+\delta)^2 - (p_x - a_x)^2 - p_y^2]^{1/2}, \tag{4}$$

where

$$a_x = b_{-1} - \frac{b_1}{2}y^2 - \frac{b_3}{4}y^4 + \cdots,$$

$$b_{-1} = \int_{s_1}^{s} b(s')ds', \quad b_1 = b', \quad b_3 = -b'''/6,$$

and

$$b(s) = \frac{1}{\rho}\frac{B_y(0,0,s)}{B_0}.$$

Now, we rewrite the Hamiltonian in terms of deviations from the design trajectory. We define vector $\mathbf{x} = (x - x^d, p_x - p_x^d, y, p_y, \tau - \tau^d, \delta)$, where superscript d refers to the design coordinates. Since the Hamiltonian does not explicitly depend on x, $p_x^d = const = \sin\psi$. We can write the full Hamiltonian to the 4^{th} order,

$$K = -[1 + 2x_6 + x_6^2 - (x_2 + \sin\psi - b_{-1} + \frac{b_1}{2}x_3^2 + \frac{b_3}{4}x_3^4)^2 - x_4^2]^{1/2}. \tag{5}$$

Defining

$$g = \sin\psi - b_{-1}, \quad n = \sqrt{1 - g^2},$$

we obtain,

$$K = -n[1 - V_1 - V_2 - V_3 - V_4]^{1/2}, \tag{6}$$

where $V_1 = \frac{2}{n^2}(gx_2 - x_6)$, $V_2 = \frac{1}{n^2}(x_2^2 + gb_1x_3^2 + x_4^2 - x_6^2)$, $V_3 = \frac{1}{n^2}b_1x_2x_3^2$, $V_4 = \frac{1}{2n^2}(gb_3 + \frac{b_1^2}{2})x_3^4$. Expanding (6), we get the Hamiltonian $K(\mathbf{x}, s)$ for the evolution of $\mathbf{x}$ [7],

$$K = K_2 + K_3 + K_4 + \cdots, \tag{7}$$

where each K_n is an nth degree polynomial in $\mathbf{x}$.

2.2.2 Calculation of M

The linear map M is found from the quadratic part of the Hamiltonian K,

$$K_2 = \frac{1}{2n}(\frac{1}{n^2}x_2^2 + gb_1x_3^2 - \frac{2g}{n^2}x_2x_6 + x_4^2 + \frac{g}{n^2}x_6^2). \tag{8}$$

Following [7] we obtain the equations for M_{ij} which together with the initial conditions give the following solution matrix,

$$M = \begin{pmatrix} 1 & M_{12} & 0 & 0 & 0 & M_{16} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_{33} & M_{34} & 0 & 0 \\ 0 & 0 & M_{43} & M_{44} & 0 & 0 \\ 0 & M_{52} & 0 & 0 & 1 & M_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{9}$$

where

$$M_{12} = \int_{s_1}^{s} \frac{1}{n^3(s')}ds', \tag{10}$$

$$M_{16} = M_{52} = M_{56} = \int_{s_1}^{s} \frac{g(s')}{n^3(s')}ds'. \tag{11}$$

The remaining matrix elements, M_{33}, M_{34}, M_{43}, M_{44}, are given by,

$$M_{34} = -\int_{s_1}^{s} \frac{1}{n(s')}M_{44}(s')ds' \tag{12}$$

and

$$M_{43} = -\int_{s_1}^{s} \frac{g(s')b_1(s')}{n(s')}M_{33}(s')ds'. \tag{13}$$

We can write the following equations for M_{33} and M_{44},

$$M'_{33}(s) = -\frac{1}{n(s)}\int_{s_1}^{s} \frac{g(s')b_1(s')}{n(s')}M_{33}(s')ds', \tag{14}$$

$$M'_{44}(s) = -\frac{g(s)b_1(s)}{n(s)}\int_{s_1}^{s} \frac{1}{n(s')}M_{44}(s')ds'. \tag{15}$$

Equations (14), (15) can be solved by iteration,

$$M_{33}(s) = 1 - \int_{s_1}^{s}\int_{s_1}^{s'} \frac{g(s'')b_1(s'')}{n(s')n(s'')}ds''ds' +$$
$$\int_{s_1}^{s}\int_{s_1}^{s'}\int_{s_1}^{s''}\int_{s_1}^{s'''} \frac{g(s'')b_1(s'')g(s'''')b_1(s'''')}{n(s')n(s'')n(s''')n(s'''')}ds''''ds'''ds''ds' + \cdots \tag{16}$$

$$M_{44}(s) = 1 - \int_{s_1}^{s}\int_{s_1}^{s'} \frac{g(s')b_1(s')}{n(s')n(s'')}ds''ds' +$$
$$\int_{s_1}^{s}\int_{s_1}^{s'}\int_{s_1}^{s''}\int_{s_1}^{s'''} \frac{g(s')b_1(s')g(s''')b_1(s''')}{n(s')n(s'')n(s''')n(s'''')}ds''''ds'''ds''ds' + \cdots \tag{17}$$

M_{34} and M_{43} are then given by (12), (13).

It should be remarked that the iterated solutions (16), (17) are just the power series expansions in the "fringe extent parameter" ϵ.

2.2.3 Calculation of f_3 and f_4

Lie Algebraic polynomials f_3 and f_4 are determined by the linear map M and the non-quadratic terms in the Hamiltonian K [7],

$$f_3(s) = -\int_{s_1}^{s} K_3(M(s')\mathbf{x})ds', \tag{18}$$

$$f_4(s) = -\int_{s_1}^{s} K_4(M(s')\mathbf{x})ds' - \frac{1}{2}\int_{s_1}^{s} :f_3(s'):K_3(M(s')\mathbf{x})ds'. \tag{19}$$

We write,

$$f_3(s_2) = \sum_{i=1}^{6}\sum_{j=1}^{i}\tau_{ij}(s_2)x_ix_j, \tag{20}$$

$$f_4(s_2) = \sum_{i=1}^{6}\sum_{j=1}^{i}\sum_{k=1}^{j}\nu_{ijk}(s_2)x_ix_jx_k, \tag{21}$$

where $\tau_{ij}(s_2)$ and $\nu_{ijk}(s_2)$ are some line integrals given in [9]; there are 10 non-zero τ's and 19 non-zero ν's.

2.2.4 Canonical Matrix Representation

We can expand exponentials in (3) to obtain a power series,

$$\begin{aligned} x_a^f &= \ldots exp(:f_4:)exp(:f_3:)M_{ab}x_b \\ &= \ldots(1 + :f_4: + \cdots)(1 + :f_3: + \frac{1}{2}:f_3:^2 + \cdots)M_{ab}x_b \\ &= M_{ab}x_b + :f_3:M_{ab}x_b + (:f_4: + \frac{1}{2}:f_3:^2)M_{ab}x_b + \cdots \end{aligned} \tag{22}$$

We can also write a formal Taylor expansion,

$$x_a^f = \mathcal{M}x_a = M_{ab}x_b + Q_{abc}x_bx_c + W_{abcd}x_bx_cx_d + \cdots, \tag{23}$$

where we sum over the repeated indices and take $d \le c \le b = 1, \ldots, 6$ to avoid the occurence of the same terms in the sum.

The terms in (22) can be identified with the matrices of (23),

$$:f_3:M_{ab}x_b \longleftrightarrow Q_{abc}x_bx_c, \tag{24a}$$

$$(:f_4: + \frac{1}{2}:f_3:^2)M_{ab}x_b \longleftrightarrow W_{abcd}x_bx_cx_d. \tag{24b}$$

There are 72 non-zero matrix elements out of total 498 (6×83): 12 M_{ab}'s, 20 Q_{abc}'s, and 40 W_{abcd}'s. They depend on τ_{ij}'s and ν_{ijk}'s. Q and W are given in [9].

2.2.5 Transformation to TRANSPORT coordinates

Let **X** denote the TRANSPORT coordinates, $\mathbf{X} = (x - x^d, x' - x'^d, y, y', L - L^d, \delta)$. We can relate X_i's to the canonical variables as follows [6],

$$x' = \frac{p_x - a_x}{\sqrt{(1+\delta)^2 - (p_x - a_x)^2 - p_y^2}}, \tag{25a}$$

$$y' = \frac{p_y}{\sqrt{(1+\delta)^2 - (p_x - a_x)^2 - p_y^2}}. \tag{25b}$$

These expressions may be inverted by solving for p_x and p_y,

$$p_x = a_x + \frac{(1+\delta)x'}{\sqrt{1 + x'^2 + y'^2}}, \tag{26a}$$

$$p_y = \frac{(1+\delta)y'}{\sqrt{1 + x'^2 + y'^2}}. \tag{26b}$$

We now evaluate equations (25), (26) at the fringe region's final and initial points, s_1 and s_2 respectively.

1. At the final point s_2:

$$B_y = B_0, \;\; b_1 = b_3 = 0, \;\; a_x(s_2) = b_{-1}(s_2) = \int_{s_1}^{s_2} b(s')ds'.$$

Recalling that $p_x = x_2 + \sin\psi$, $g = \sin\psi - b_{-1}$ and denoting coordinates at s_2 with the superscript f, we get,

$$X_1^f = x_1^f, \tag{27a}$$

$$X_2^f = \frac{g}{n} + \frac{x_2^f + g}{\sqrt{(1 + x_6^f)^2 - (x_2^f + g)^2 - (x_4^f)^2}}, \tag{27b}$$

$$X_3^f = x_3^f, \tag{27c}$$

$$X_4^f = \frac{x_4^f}{\sqrt{(1 + x_6^f)^2 - (x_2^f + g)^2 - (x_4^f)^2}}. \tag{27d}$$

2. At the initial point s_1:

$$B = 0, \;\; a_x(s_1) = 0.$$

Remembering that $x'^d = \tan\psi$ initially, we get

$$x_1 = X_1, \tag{28a}$$

$$x_2 = \sin\psi + \frac{(1 + X_6)(X_2 + \tan\psi)}{\sqrt{1 + (X_2 + \tan\psi)^2 + X_4^2}}, \tag{28b}$$

$$x_3 = X_3, \tag{28c}$$

$$x_4 = -\frac{(1 + X_6)X_4}{\sqrt{1 + (X_2 + \tan\psi)^2 + X_4^2}}. \tag{28d}$$

2.2.6 TRANSPORT transfer matrices

Given M, Q, W of (23), R, T, U of (1) are obtained as follows,

1. Expand (27) in a power series to obtain,

$$X_a^f = L_{ab}x_b^f + N_{abc}x_b^f x_c^f + P_{abcd}x_b^f x_c^f x_d^f + \cdots \tag{29}$$

2. Substitute the power expansion (23) for x^f in (29) to obtain,

$$X_a^f = \mathcal{L}_{ab}x_b + \mathcal{N}_{abc}x_b x_c + \mathcal{P}_{abcd}x_b x_c x_d + \cdots \tag{30}$$

3. Expand (28) in a power series to obtain,

$$x_a = \tilde{L}_{ab}X_b + \tilde{N}_{abc}X_b X_c + \tilde{P}_{abcd}X_b X_c X_d + \cdots \tag{31}$$

and substitute (31) into (30) to obtain the desired R_{ab}, T_{abc}, U_{abcd}.

The details of the calculations and the final matrices are found in [9].

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