NONLINEAR EFFECTS IN HIGH-CURRENT BEAMS

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Abstract: Within perturbation theory the formulas are derived expressing the time dependence of emittance growth in an arbitrary linear focusing system through the Fourier coefficients of the additional nonlinear part of the Hamiltonian. These coefficients are calculated for space-charge potential of a beam with elliptical symmetry. As an example, the expression is obtained for the Coulomb emittance growth of the beam with waterbag distribution in a short focusing system.

Emittance growth due to nonlinearity

The basic equations were derived by Wangler et al. /1/ and Hofmann /2/ relating rms emittance growth of a beam to its electric field energy. However, these equations don't allow to study the time evolution of the process. This time evolution was considered by Anderson /3/ for particular case of the sheet beam with laminar motion in a constant focusing channel.

Our new approach within perturbation theory investigates the internal motion of the beam and from this obtains the explicit time dependence of the rms emittance. Analysis is performed for two-dimensional beams with elliptical symmetry by introducing of the next simplifications: 1) perturbations of the particle trajectories are small; 2) the additional electric fields due to the coherent oscillations are not taken into account.

We start our analysis with well-known Sacherer's /4/ differential equations for rms envelopes as functions of the distance z along the axis of a channel:

$$(b_{x,y})'' + \tilde{K}_{x,y}(z)\, b_{x,y} - \frac{\mathcal{E}_{x,y}^2}{b_{x,y}^3} - \frac{K}{2(b_x + b_y)} = 0, \quad (1)$$

where K is perveance; $b_x = \sqrt{\langle x^2 \rangle}$; $b_y = \sqrt{\langle y^2 \rangle}$; x and y are transverse coordinates; sign $\langle\ \rangle$ denotes the averaging on the particle distribution function; $\mathcal{E}_x = \mathcal{E}_y = \mathcal{E}$ is rms emittance:

$$\mathcal{E}^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle x x' \rangle^2 \quad . \quad (2)$$

The corresponding equations for the linear transverse oscillations of the particles have the form (in the x plane):

$$x'' + \tilde{K}_x(z)\, x - \frac{K}{2 b_x (b_x + b_y)}\, x = 0 \quad . \quad (3)$$

The solution of the system (1) is determined by the initial values of $b_{x,y}$ and their derivatives. Substituting this solution into Eqs.(3), we are able to find the fundamental solutions Ψ and $\overset{*}{\Psi}$ (sign $*$ indicates the complex conjugation) which determine the motion of individual particles.

For the standart normalizing condition ($\Psi \overset{*}{\Psi}' - \overset{*}{\Psi}\Psi' = -2iW$, where W is some positive constant) the amplitude and phase of the fundamental solutions are connected with rms envelopes by the next formulas:

$$|\Psi|_{x,y} = b_{x,y}\sqrt{\frac{W}{\mathcal{E}}} \quad ; \quad \alpha_{x,y} = \arg \Psi_{x,y} = \int_0^z \frac{\mathcal{E}}{b_{x,y}^2}\, d\tilde{z} \quad . \quad (4)$$

Expressing the coordinates of a single particle through the fundamental solutions, we obtain:

$$x = C_x \Psi_x + \overset{*}{C_x}\overset{*}{\Psi_x} = \sqrt{\frac{2 J_x}{\mathcal{E}}}\, b_x \cos \Phi_x \quad , \quad (5)$$

where C_x is the complex amplitude of the oscillations; $J_x = 2W|C_x|^2$; $\Phi_x = \alpha_x + \chi_x$; $\chi_x = \arg C_x$.

The real electromagnetic field acting on the particles of a beam differs from the linear fields which are included into the system (3). This additional field corresponds to the additional part $\Delta H(x,y,z)$ of the Hamiltonian.

With the help of Eq.(5) we introduce the canonical variables $\vec{J} - \vec{\chi}$ (action-phase) and expand ΔH into the Fourier series on $\vec{\Phi}$ (arrow denots vector):

$$\Delta H(\vec{J}, \vec{\Phi}, z) = \sum_{\vec{n}} G_{\vec{n}}(\vec{J}, z) \exp(i\vec{n}\vec{\Phi}) \quad , \quad (6)$$

where $G_{\vec{n}}(\vec{J}, z) = \frac{1}{(2\pi)^2} \int_0^{2\pi}\int_0^{2\pi} \Delta H \exp(i\vec{n}\vec{\Phi})\, d\Phi_x\, d\Phi_y \quad . \quad (7)$

Using the Hamilton's equations for the variables $\vec{J}$ and $\vec{\chi}$:

$$\vec{J}' = -\frac{\partial(\Delta H)}{\partial \vec{\chi}} \quad ; \quad \vec{\chi}' = \frac{\partial(\Delta H)}{\partial \vec{J}} \quad , \quad (8)$$

we derive the next system of the equations:

$$\vec{\mathcal{J}}' = -i \sum_{\vec{n}} \vec{n}\, G_{\vec{n}}\,(\vec{\mathcal{J}},\varkappa)\, exp\,(i\vec{n}\,\vec{\Phi})$$

$$\vec{\chi}' = \sum_{\vec{n}} \frac{\partial G_{\vec{n}}}{\partial \vec{\mathcal{J}}}\, exp\,(i\vec{n}\,\vec{\Phi}) \qquad (9)$$

The solution of the system (9) can be found by use of the successive approximations method:

$$\vec{\mathcal{J}} = \vec{I} + \vec{I}_1 + \vec{I}_2 + \dots \;\; ; \;\; \vec{\chi} = \vec{\chi}_o + \vec{\chi}_1 + \vec{\chi}_2 + \dots \qquad (10)$$

where $\vec{I}$ and $\vec{\chi}_o$ are zero-order approximations; $\vec{I}_1$ and $\vec{\chi}_1$ are first-order approximations etc.

Let us take $\vec{I}$ equal to the initial (unperturbed) value when $\varkappa = 0$ and $\vec{\chi}_o$ equal to

$$\vec{\chi}_o = \vec{\chi}_{in} + \vec{A} \quad , \qquad (11)$$

where $\vec{A}(\vec{I},\varkappa) = \int_0^\varkappa \frac{\partial G_{o,o}}{\partial \vec{\mathcal{J}}}(\vec{I},\tilde{\varkappa})\, d\tilde{\varkappa}$;

$\vec{\chi}_{in}$ is the initial value of the phase.

We calculate iterations (10) in Ref./5/ and obtain the expression for rms emittance growth of a beam with uniform distribution on the initial phases.

In the first approximation $\Delta \mathcal{E}^2 = 0$ because $\langle \vec{I}_1 \rangle = \langle \vec{\chi}_1 \rangle = 0$.

In the second approximation we derive the next general formulas for the dependence of the emittance growth on the Fourier coefficients of the additional part of the Hamiltonian:

$$\Delta \mathcal{E}^2_{x,y} = 2\langle I_{x,y}\rangle\langle (I_2)_{x,y}\rangle - |\Gamma'_{x,y}|^2 , \qquad (12)$$

where $\langle \vec{I}_2 \rangle = \sum_{\vec{n}} \vec{n} \langle Re[\, B_{\vec{n}}(\vec{I},\varkappa)\vec{n}\,\overset{*}{\mathcal{D}}_{\vec{n}}(\vec{I},\varkappa)]\rangle$; (13)

$$\Gamma'_x = 2\langle B_{-2,0}(\vec{I},\varkappa) + I_x [\,\mathcal{D}_{-2,0}(\vec{I},\varkappa)]_x \rangle ; \qquad (14)$$

$$B_{\vec{n}} = \int_0^\varkappa G_{\vec{n}}(\vec{I},\tilde{\varkappa}) exp\,[i\vec{n}(\vec{\varkappa}+\vec{A})]\, d\tilde{\varkappa} ; \qquad (15)$$

$$\mathcal{D}_{\vec{n}} = \int_0^\varkappa \frac{\partial G_{\vec{n}}}{\partial \vec{\mathcal{J}}}(\vec{I},\tilde{\varkappa}) exp\,[i\vec{n}(\vec{\varkappa}+\vec{A})]\, d\tilde{\varkappa} . \qquad (16)$$

Emittance growth due to space-charge nonlinearity

The additional part of the Hamiltonian can be obtained by solving of Poisson's equation for a beam with rms envelopes defining by Eqs.(1).

The density in real space is

$$\rho(x,y,\varkappa) = \rho_o F(\frac{x^2}{a_x^2} + \frac{y^2}{a_y^2}) = \rho_o F(T) , \qquad (17)$$

where ρ_o is the charge density in the beam centre; $a_{x,y}$ are the beam envelopes; the function F satisfies to the next conditions: $F(0)=1$; $F(1)=0$.

The power series expansion of F has a form:

$$F(T) = \sum_{\ell=0}^{\infty} \mathcal{V}_\ell (\frac{x^2}{a_x^2} + \frac{y^2}{a_y^2})^\ell . \qquad (18)$$

The electrostatic field inside the beam with elliptical symmetry of the charge density is /4/

$$\tilde{E}_x = 2\pi\rho_o a_x a_y x \int_0^\infty \frac{F(\frac{x^2}{a_x^2+S} + \frac{y^2}{a_y^2+S})}{\sqrt{(a_x^2+S)^3(a_y^2+S)}}\, dS . \qquad (19)$$

Substitution of Eq.(18) into (19) and integration of Eq.(19) yield the additional Coulomb part of the Hamiltonian. The amplitudes of the Fourier harmonics are determined as follows /5/:

$$G_{2\vec{n}}(\vec{\mathcal{J}},\varkappa) = K \sum_{\vec{m}} R_{\vec{m}} \Lambda_{\vec{n}}^{\vec{m}} \mathcal{J}_x^{m_x} \mathcal{J}_y^{m_y} , \qquad (20)$$

where $R_{\vec{m}}(\theta_x,\theta_y) = C_{m_x+m_y}^{m_y} \mathcal{V}_{m_x,m_y} \theta_x^{2m_x} \theta_y^{2m_y}$; (21)

$$\Lambda_{\vec{n}}^{\vec{m}} = -\frac{\sigma_1}{2} \frac{\mathcal{V}_{m_x+m_y-1}}{(4 I_o)^{m_x+m_y}(m_x+m_y)} C_{2m_x}^{m_x-n_x} C_{2m_y}^{m_y-n_y} ; \qquad (22)$$

$$\frac{1}{\sigma_1} = \int_0^1 F(T)\, dT ; \qquad (23)$$

C_ℓ^j are binomial coefficients; $I_o = max(\mathcal{J}_{x,y})$.

The values $V_{\ell,j}(\theta_x,\theta_y)$ can be calculated by use of the recurrence relations:

$$V_{1,0} = \frac{2}{\theta_x(\theta_x+\theta_y)} ; \quad V_{0,1} = \frac{2}{\theta_y(\theta_x+\theta_y)} ;$$

$$\qquad (24)$$

$$V_{\ell+1,j} = -\frac{1}{(2\ell+1)\theta_x}\frac{\partial V_{\ell,j}}{\partial \theta_x} ; \quad V_{\ell,j+1} = -\frac{1}{(2j+1)\theta_y}\frac{\partial V_{\ell,j}}{\partial \theta_y} .$$

Substitution of the space-charge Fourier harmonics into the general formulas (12)-(16) provides rather unwieldy expressions for the

emittance growth of a beam with arbitrary distribution in phase space.

Here we restrict ourselves by the relatively simple case of waterbag distribution. Then for a short focusing channel with the small nonlinear phase advance ($|\vec{A}(\vec{I},\varkappa)| \ll 1$) we derive:

$$\Delta \mathcal{E}_x^2(\varkappa) = \frac{K^2}{960} \left\{ |E_{1,0}^{2,0}|^2 - Re\,(E_{1,0}^{2,0}\overset{*}{E}_{1,0}^{1,1}) + \frac{1}{4}|E_{1,0}^{1,1}|^2 + \right.$$
$$\left. + 4|E_{2,0}^{2,0}|^2 + \frac{1}{3}|E_{1,1}^{1,1}|^2 \right\}, \qquad (25)$$

where $E_n^{\vec{m}}$ are the universal structure integrals:

$$E_{1,0}^{2,0} = \int_0^\varkappa R_{2,0} \exp(i2\alpha_x)\,d\tilde{\varkappa}; \quad E_{1,0}^{1,1} = \int_0^\varkappa R_{1,1} \exp(i2\alpha_x)\,d\tilde{\varkappa};$$
$$E_{2,0}^{2,0} = \int_0^\varkappa R_{2,0} \exp(i4\alpha_x)\,d\tilde{\varkappa}; \quad E_{1,1}^{1,1} = \int_0^\varkappa R_{1,1} \exp[i2(\alpha_x+\alpha_y)]\,d\tilde{\varkappa}; \qquad (26)$$

values $R_{\vec{m}}$ equal to

$$R_{2,0} = \frac{2\,6_x}{3}\,\frac{2\,6_x+6_y}{(6_x+6_y)^2}; \quad R_{1,1} = \frac{4\,6_x 6_y}{(6_x+6_y)^2} \qquad (27)$$

($\Delta \mathcal{E}_y^2$ can be obtained by replacing of indexes).
For a round beam Eq.(25) becomes:

$$\Delta \mathcal{E}^2(\varkappa) = \frac{K^2}{720}\,|E|^2, \qquad (28)$$

where $\quad E = \int_0^\varkappa \exp(i4\alpha_{x,y})\,d\tilde{\varkappa}$.

In the focusing systems with the small phase advance

$$\Delta \mathcal{E}^2 \approx 0,00139\,(K\varkappa)^2. \qquad (29)$$

According to the Anderson's theory /3/ for a round beam, it can be found for small $\omega \varkappa$ (to η^2 order in expression for $\tau(\rho,\varkappa)$ in notation of Ref./3/):

$$\Delta \mathcal{E}^2 \approx 0,00137\,(K\varkappa)^2. \qquad (29)'$$

The Eqs.(29) and (29)' coincide with a good accuracy.

The results obtained here are valid only for $\frac{\Delta \mathcal{E}}{\mathcal{E}} \ll 1$.

In the space-charge-dominated beams the density profile undergoes significant changes in about one quarter of a plasma period /1/. Thus, for such beams our theory is correct if

$$\int_0^\varkappa \omega_p\,d\tilde{\varkappa} < \frac{\pi}{2} \quad , \text{ where } \quad \omega_p^2 = \frac{2K}{a_x a_y} . \qquad (30)$$

References

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