MOMENT EQUATIONS WITH SPACE CHARGE FOR AN AVF CYCLOTRON

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Abstract

A set of differential equations is derived for the time-evolution under space charge conditions, of the second order moments of the phase-space distribution of a bunched beam. The model takes into account two special features of an AVF cyclotron namely dispersion (i.e. radial-longitudinal coupling) and isochronism. The method used can also be applied to other types of circular accelerators. It is based on the RMS approach; it assumes linear space charge forces (determined by a least squares method) and it assumes an ellipsoidal charge distribution which may be rotated around its vertical axis. Several integrals of motion are given such as the total angular canonical momentum in the bunch, the total energy-content of the bunch and the RMS-representation of the 4-dimensional horizontal phase-space volume.

1. Introduction

The thus far published analytical studies of space charge effects mainly deal with linear accelerator structures (1-3). For circular accelerators like the cyclotron the analysis is mostly done with numerical many-particle codes (4,5). Circular accelerators have the special feature that, due to dispersion, the radial particle position depends on the longitudinal momentum. This means that particles in the "tail" of the bunch, who lose energy due to the longitudinal space charge force, will move to a lower radius. The opposite can happen for the leading particles. For the isochronous cyclotron there is a second special feature namely that there is no longitudinal RF-focussing to counteract the space charge forces. In this situation the longitudinal-transverse coupling (dispersion) may become extra important (4,6).

We represent the bunch-properties by second moments of the phase-space distribution and derive a set of differential equations for their time evolution under space charge conditions. The derivation uses the Hamiltonian formalism and is an application of the RMS-(Root Mean Square)approach, assuming linear space charge forces as determined by a least squares method and assuming an ellipsoidal charge distribution (2). We allow the ellipsoid to be rotated around its vertical axis. The method used, can be applied to all types of circular accelerators. We consider as a special case the isochronous cyclotron for which a smoothing procedure will be applied to eliminate the time dependence resulting from the azimuthally varying guide field. For convenience we omit acceleration effects. The derivation is such however, that these can be included by using methods developed by Schulte et. al. (7,8).

2. The Hamiltonian without space charge

We describe the motion of the particles in a coordinate system which moves with the bunch along the equilibrium orbit of a reference particle having a kinetic momentum p_o.

We approximate this motion with linear equations. Then, the Hamiltonian is quadratic in the variables. For a circular accelerator the Hamiltonian, in the absence of acceleration, becomes (9):

$$\bar{H}_o = \frac{1}{2}\bar{p}_x^2 + \frac{1}{2}\bar{p}_s^2 + \frac{1}{2}\bar{p}_z^2 + \frac{1}{2}Q_x(\tau)\bar{x}^2 + \frac{1}{2}Q_z(\tau)\bar{z}^2 - \gamma\eta(\tau)\bar{x}\bar{p}_s \quad (1)$$

where $\bar{x}$, $\bar{s}$, $\bar{z}$ are the coordinates, $\bar{p}_x$, $\bar{p}_s$, $\bar{p}_z$ the canonical momenta and τ the independent variable. The variables and also the Hamiltonian are scaled quantities:

$$\bar{x} = x/R_o \qquad \bar{z} = z/R_o \qquad \bar{s} = \gamma s/R_o$$
$$\bar{p}_x = p_x/p_o \qquad \bar{p}_z = p_z/p_o \qquad \bar{p}_s = p_s/(\gamma p_o) \quad (2)$$
$$\tau = v_o t/R_o \qquad \bar{H}_o = H_o/(v_o p_o)$$

Here R_o is the effective radius of the equilibrium orbit, v_o the velocity of the reference particle and γ the relativistic factor $(1-v_o^2/c^2)^{-\frac{1}{2}}$. We note that the scaling of the longitudinal variables involves this γ. The non-scaled coordinates x, s and z represent the horizontal, longitudinal and vertical distance in the bunch. The variables p_x and p_z give the transverse components of the kinetic momentum of the particle. The variable p_s gives the deviation between the longitudinal kinetic momentum of the particle and p_o. The time unit τ is defined such that an increase of 2π corresponds with one revolution of the bunch. The quantities $Q_x(\tau)$ and $Q_z(\tau)$ represent the azimuthally varying horizontal and vertical focussing strength of the magnetic guide field. The quantity $\eta(\tau)$ represents the bending strength of the guide field. It is inversely proportional to the local radius of curvature of the equilibrium orbit. For an AVF cyclotron the quantities Q_x, Q_z and η can be expressed in terms of the Fourier components of the magnetic field and their radial derivatives (9). The equations of motion derived from the Hamiltonian $\bar{H}_o$ give the well-known result that the vertical coordinate obeys an homogeneous Hill-equation and the horizontal coordinate an inhomogeneous Hill-equation with momentum deviation as the driving term.

Due to the azimuthal variation of the magnetic guide field the quantities Q_x, Q_z and η in Eq. (1) are "time-dependent". As has been shown in Ref. (10) the oscillating (i.e. time-dependent) parts of a Hamiltonian can be transformed to higher order in the flutter (i.e. the azimuthally varying part of the magnetic field) such that, within the approximations used, the new oscillating parts can be neglected. For this purpose a linear canonical transformation (to new variables $\tilde{x}$, $\tilde{s}$, $\tilde{z}$, $\tilde{p}_x$, $\tilde{p}_s$, $\tilde{p}_z$) is applied which changes coordinates and momenta only slightly; the difference between the old and new variables being of the order of the flutter. The new smoothed Hamiltonian $\tilde{H}_o$ has the same shape as in Eq. (1) but with Q_x, Q_z and η replaced by time-independent quantities ν_x^2, ν_z^2 and $\bar{\eta}$ respectively. Here ν_x and ν_z are the horizontal and vertical tune respectively and $\bar{\eta}$ is the "average bending strength" of the guide field. For an AVF cyclotron the quantities ν_x, ν_z and $\bar{\eta}$ can again be expressed in terms of the Fourier components of the flutter (9). We note that ν_x and ν_z are also given in Ref. (10).

If we assume that the cyclotron is perfectly isochronous, then we can establish a relation between the quantities γ, $\bar{\eta}$ and ν_x. This relation can be found from the equations of motion derived from $\tilde{H}_o$ by using the fact that $d\tilde{s}/d\tau$ must be zero for particles which have a deviating momentum $\tilde{p}_s$ and move on the corresponding equilibrium orbit. The relation is given by:

$$\gamma^2 - (\nu_x/\bar{\eta})^2 = 0 \quad (3)$$

By using this equation we can remove the quantities γ and $\bar{\eta}$ in $\tilde{H}_o$ in favour of ν_x. This then gives the smoothed Hamiltonian for an isochronous cyclotron. In order to bring this Hamiltonian in a symmetric form (with respect to the longitudinal and horizontal variables) we apply a final canonical transformation. This transformation leaves the vertical variables unchanged ($\hat{z} = \tilde{z}$, $\hat{p}_z = \tilde{p}_z$) and also the coordinates $\tilde{x}$ and $\tilde{s}$ ($\hat{x} = \tilde{x}$, $\hat{s} = \tilde{s}$). The new momenta $\hat{p}_x$ and $\hat{p}_z$ are defined as:

$$\hat{p}_x = \tilde{p}_x - \frac{1}{2}\nu_x\tilde{s} \quad , \quad \hat{p}_s = \tilde{p}_s - \frac{1}{2}\nu_x\tilde{x} \tag{4}$$

We note that these new momenta are no longer equal to the kinetic momenta. By using Eq. (1) (with $Q_x = \nu_x^2$, $Q_z = \nu_z^2$, $\eta = \bar{\eta}$) and Eqs. (3) and (4) the smoothed Hamiltonian in the new variables can be written as:

$$\hat{H}_o = \frac{1}{2}(\hat{p}_x + \frac{1}{2}\nu_x\hat{s})^2 + \frac{1}{2}(\hat{p}_s - \frac{1}{2}\nu_x\hat{x})^2 + \frac{1}{2}\hat{p}_z^2 + \frac{1}{2}\nu_z^2\hat{z}^2 \tag{5}$$

As a remark we note that the horizontal part of $\hat{H}_o$ has the same structure as the Hamiltonian for a particle that moves in a homogeneous magnetic field. In section 4 we will use Eq. (5) to derive the smoothed moment equations for an isochronous cyclotron.

3. The space charge potential function

The Hamiltonian under space charge conditions $\bar{H}$ is found by adding to the unperturbed Hamiltonian H a (scaled) electric potential function $\bar{\phi} = q\phi/(v_o p_o)$ which represents the self-field of the bunch. We asume that the radial size of the bunch is much smaller than the local radius of curvature. For the calculation of $\bar{\phi}$ the curved coordinate system may then be approximated to be cartesian. With the scaling of Eqs. (2) the Poisson equation becomes:

$$\frac{\partial^2\bar{\phi}}{\partial\bar{x}^2} + \frac{\partial^2\bar{\phi}}{\partial\bar{s}^2} + \frac{\partial^2\bar{\phi}}{\partial\bar{z}^2} = -\left(\frac{q^2}{\epsilon_o\gamma\, v_o R_o p_o}\right)\bar{\rho} \tag{6}$$

where $\bar{\rho} = R^3\rho/(q\gamma)$ is the (scaled) charge density. Equation (6) also includes the magnetic self-field of the bunch.

We want to use a linear approximation for the space charge forces. A General shape for the potential function then becomes:

$$\bar{\phi}(\bar{x},\tau) = -\frac{1}{2}a(\tau)\bar{x}^2 - d(\tau)\bar{x}\bar{s} - \frac{1}{2}b(\tau)\bar{s}^2 - \frac{1}{2}c(\tau)\bar{z}^2 \tag{7}$$

Here we have taken into account (via the term $d(\tau)\bar{x}\bar{s}$) a possible non-symmetric distribution of the bunch, as may result from the transverse-longitudinal coupling in $\hat{H}_o$. For the definition of a, b, c and d we use the least squares method as introduced in Ref. (2), i.e. we minimize the averaged difference D between the true electric field $\bar{E}$ and its linear approximation $\bar{E}_o$ which follows from Eq. (7). D is given by:

$$D = \int_{-\infty}^{\infty} |\bar{E} - \bar{E}_o|^2\, \bar{\rho}\,(\bar{x},\tau)d\bar{x} \tag{8}$$

We derive $\bar{E}_o$ from Eq. (7) ($\bar{E}_o = -\nabla\bar{\phi}$) and substitute this in Eq. (8). Then we differentiate Eq. (8) to a, b, c and d. This gives a set of four equations for a, b, c and d. Its solution gives a, b, c and d in terms of the second moments $\langle \bar{x}^2 \rangle$, $\langle \bar{x}\bar{s} \rangle$, $\langle \bar{s}^2 \rangle$ and $\langle \bar{z}^2 \rangle$ and five unknown terms namely $\langle \bar{x}\bar{E}_x \rangle$, $\langle \bar{s}\bar{E}_x \rangle$, $\langle \bar{x}\bar{E}_s \rangle$, $\langle \bar{s}\bar{E}_s \rangle$ and $\langle \bar{z}\bar{E}_z \rangle$. In the following we express these terms as functions of $\langle \bar{x}^2 \rangle$, $\langle \bar{x}\bar{s} \rangle$, $\langle \bar{s}^2 \rangle$ and $\langle \bar{z}^2 \rangle$. For this we assume that the charge distribution has ellipsoidal symmetry, i.e.:

$$\bar{\rho} = \bar{\rho}(U),\ U = (\bar{x}/\bar{A})^2 + 2\bar{x}\bar{s}/\bar{D}^2 + (\bar{s}/\bar{B})^2 + (\bar{z}/\bar{C})^2 \tag{9}$$

The corresponding electric field follows from the Poisson equation Eq. (6). In order to find the solution we rotate the coordinate frame over a (yet unknown) angle φ such that in the new frame $\bar{\rho}$ takes the simpler form:

$$\bar{\rho} = \bar{\rho}\,[(\tilde{x}/A)^2 + (\tilde{s}/B)^2 + (\tilde{z}/C)^2] \tag{10}$$

(Note that the rotated coordinates $\tilde{x}$, $\tilde{s}$, $\tilde{z}$ should not be mixed up with the variables introduced in section 2.) To find expressions for A, B, C and φ we calculate the new second moments of the charge distribution Eq. (10). These are related with the old moments via the coordinate rotation. The angle φ is determined by the requirement that in the rotated frame $\langle \tilde{x}\tilde{s} \rangle$ must vanish. We find for A, B, C and φ:

$$\begin{aligned}(A/k)^2 &= \tfrac{1}{2}[\langle \bar{x}^2 \rangle(1+1/\cos2\varphi) + \langle \bar{s}^2 \rangle(1-1/\cos2\varphi)]\\ (B/k)^2 &= \tfrac{1}{2}[\langle \bar{x}^2 \rangle(1-1/\cos2\varphi) + \langle \bar{s}^2 \rangle(1+1/\cos2\varphi)]\\ (C/k)^2 &= \langle \bar{z}^2 \rangle,\quad \tan2\varphi = 2\langle \bar{x}\bar{s} \rangle/(\langle\bar{s}^2\rangle - \langle\bar{x}^2\rangle)\end{aligned} \tag{11}$$

where k is a yet unimportant constant. The averages $\langle \tilde{x}\tilde{E}_x \rangle$, $\langle \tilde{s}\tilde{E}_s \rangle$ and $\langle \tilde{z}\tilde{E}_z \rangle$ result from the rotated charge distribution Eq. (10). These have been given by Sacherer (2). The averages in the non-rotated frame then follow from the inverse coordinate rotation applied on $\langle \tilde{x}\tilde{E}_x \rangle$, $\langle \tilde{s}\tilde{E}_s \rangle$ and $\langle \tilde{z}\tilde{E}_z \rangle$. The result is:

$$\begin{aligned}\langle \bar{x}\,\bar{E}_x \rangle &= \frac{I}{I_o}\left[\frac{k}{A}\,g\left(\frac{B}{A},\frac{C}{A}\right)\cos^2\varphi + \frac{k}{B}\,g\left(\frac{C}{B},\frac{A}{B}\right)\sin^2\varphi\right]\\ \langle \bar{s}\,\bar{E}_x \rangle &= -\frac{I}{I_o}\left[\frac{k}{A}\,g\left(\frac{B}{A},\frac{C}{A}\right) - \frac{k}{B}\,g\left(\frac{C}{B},\frac{A}{B}\right)\right]\sin\varphi\cos\varphi\\ \langle \bar{s}\,\bar{E}_s \rangle &= \frac{I}{I_o}\left[\frac{k}{A}\,g\left(\frac{B}{A},\frac{C}{A}\right)\sin^2\varphi + \frac{k}{B}\,g\left(\frac{C}{B},\frac{A}{B}\right)\cos^2\varphi\right]\\ \langle \bar{z}\,\bar{E}_z \rangle &= \frac{I}{I_o}\frac{k}{C}\,g\left(\frac{A}{C},\frac{B}{C}\right),\quad \langle \bar{x}\bar{E}_s \rangle = \langle \bar{s}\bar{E}_x \rangle\end{aligned} \tag{12}$$

where I is the average beam current. The function g has been defined in Ref. (2). The characteristic current I_o is defined as:

$$I_o = 2hm_o\epsilon_o c^3\,(\gamma^2-1)^{3/2}/(q\lambda_3\gamma) \tag{13}$$

where h gives the number of bunches per turn. The parameter λ_3 (as given in Ref. (2)) depends only weakly on the type of distribution chosen. We can take $\lambda_3 = 1/(5\sqrt{5})$ which corresponds with a uniform distribution. The coefficients a, b, c and d in Eq. (7) have now been specified completely in terms of second moments. The solution can be written as:

$$\begin{aligned}a &= (\langle\bar{s}^2\rangle\langle\bar{x}\bar{E}_x\rangle - \langle\bar{x}\bar{s}\rangle\,\langle\bar{s}\bar{E}_x\rangle)/(\langle\bar{x}^2\rangle\langle\bar{s}^2\rangle - \langle\bar{x}\bar{s}\rangle^2)\\ b &= (\langle\bar{x}^2\rangle\langle\bar{s}\bar{E}_s\rangle - \langle\bar{x}\bar{s}\rangle\,\langle\bar{s}\bar{E}_x\rangle)/(\langle\bar{x}^2\rangle\langle\bar{s}^2\rangle - \langle\bar{x}\bar{s}\rangle^2)\\ c &= \langle\bar{z}\bar{E}_z\rangle/\langle\bar{z}^2\rangle,\quad d = \langle\bar{x}\bar{s}\rangle\,(a-b)/(\langle\bar{x}^2\rangle - \langle\bar{s}^2\rangle)\end{aligned} \tag{14}$$

4. Moment equations with space charge

We form ten independent second order moments corresponding with the horizontal motion and three moments corresponding with the vertical motion. We note that, within our approximations, the horizontal and vertical motion are uncoupled. Therefore we do not have to consider cross-terms between horizontal and vertical variables. The Hamiltonian with space charge $\bar{H} = \hat{H}_o + \bar{\phi}$ determines the time evolution of the moments. This can be seen by considering for example the moment $\langle\bar{x}^2\rangle$. For this we have: $d\langle\bar{x}^2\rangle/d\tau = 2\langle\bar{x}d\bar{x}/d\tau\rangle = 2\langle\bar{x}\partial\bar{H}/\partial\bar{p}_x\rangle$. We repeat this for the other moments and then arrive at a non- linear system of thirteen coupled first order differential equations. This system is closed (i.e. does not contain unknown terms) because i) the equations of motion of a single particle are linear (therefore no third or higher moments are involved) and ii) the coefficients a, b, c and d are already expressed in terms of the second moments.

The system of moment equations obtained can be applied to all types of circular accelerators. We will not write down this system. Instead, we consider as a special case the isochronous cyclotron described by the Hamiltonian $\hat{H}_o$ given in Eq. (5). The smoothing transformation and the canonical transformation defined in Eqs. (4) in principle have to be applied also on the electric potential function $\bar{\phi}$. The latter is a point transformation which leaves the potential function unchanged. We note that the moments which involve $\hat{p}_x$ and $\hat{p}_s$ obtain another meaning due to this transformation. The relation between the old and new moments is determined by Eqs. (4) however. As for the smoothing procedure, the difference between the old and new variables will be small. We neglect the change of the potential function due to this transformation. The system of moment equations for the isochronous cyclotron becomes:

$$\frac{d}{d\tau}\langle\hat{x}^2\rangle = 2\langle\hat{x}\hat{p}_x\rangle + \nu_x\langle\hat{x}\hat{s}\rangle$$
$$\frac{d}{d\tau}\langle\hat{x}\hat{p}_x\rangle = \langle\hat{p}_x^2\rangle - \frac{1}{4}(\nu_x^2-4a)\langle\hat{x}^2\rangle + \frac{1}{2}\nu_x(\langle\hat{x}\hat{p}_s\rangle+\langle\hat{s}\hat{p}_x\rangle) + d\langle\hat{x}\hat{s}\rangle$$
$$\frac{d}{d\tau}\langle\hat{p}_x^2\rangle = \nu_x\langle\hat{p}_x\hat{p}_s\rangle - \frac{1}{2}(\nu_x^2-4a)\langle\hat{x}\hat{p}_x\rangle + 2d\langle\hat{s}\hat{p}_x\rangle$$

$$\frac{d}{d\tau}\langle\hat{s}^2\rangle = 2\langle\hat{s}\hat{p}_s\rangle - \nu_x\langle\hat{x}\hat{s}\rangle$$
$$\frac{d}{d\tau}\langle\hat{s}\hat{p}_s\rangle = \langle\hat{p}_s^2\rangle - \frac{1}{4}(\nu_x^2-4b)\langle\hat{s}^2\rangle - \frac{1}{2}\nu_x(\langle\hat{x}\hat{p}_s\rangle+\langle\hat{s}\hat{p}_x\rangle) + d\langle\hat{x}\hat{s}\rangle$$
$$\frac{d}{d\tau}\langle\hat{p}_s^2\rangle = -\nu_x\langle\hat{p}_x\hat{p}_s\rangle - \frac{1}{2}(\nu_x^2-4b)\langle\hat{s}\hat{p}_s\rangle + 2d\langle\hat{x}\hat{p}_s\rangle$$

$$\frac{d}{d\tau}\langle\hat{x}\hat{s}\rangle = \frac{1}{2}\nu_x(\langle\hat{s}^2\rangle-\langle\hat{x}^2\rangle) + \langle\hat{x}\hat{p}_s\rangle + \langle\hat{s}\hat{p}_x\rangle$$
$$\frac{d}{d\tau}\langle\hat{p}_x\hat{p}_s\rangle = \frac{1}{2}\nu_x(\langle\hat{p}_s^2\rangle-\langle\hat{p}_x^2\rangle) - \frac{1}{4}(\nu_x^2-4a)\langle\hat{x}\hat{p}_s\rangle + $$
$$- \frac{1}{4}(\nu_x^2-4b)\langle\hat{s}\hat{p}_x\rangle + d(\langle\hat{x}\hat{p}_x\rangle+\langle\hat{s}\hat{p}_s\rangle)$$
$$\frac{d}{d\tau}\langle\hat{x}\hat{p}_s\rangle = \langle\hat{p}_x\hat{p}_s\rangle + \frac{1}{2}\nu_x(\langle\hat{s}\hat{p}_s\rangle-\langle\hat{x}\hat{p}_x\rangle) + \frac{1}{4}(\nu_x^2-4b)\langle\hat{x}\hat{s}\rangle + d\langle\hat{x}^2\rangle$$
$$\frac{d}{d\tau}\langle\hat{s}\hat{p}_x\rangle = \langle\hat{p}_x\hat{p}_s\rangle + \frac{1}{2}\nu_x(\langle\hat{s}\hat{p}_s\rangle-\langle\hat{x}\hat{p}_x\rangle) + \frac{1}{4}(\nu_x^2-4a)\langle\hat{x}\hat{s}\rangle + d\langle\hat{s}^2\rangle$$

$$\frac{d}{d\tau}\langle\hat{z}^2\rangle = 2\langle\hat{z}\hat{P}_z\rangle$$
$$\frac{d}{d\tau}\langle\hat{z}\hat{p}_z\rangle = \langle\hat{p}_z^2\rangle - (\nu_z^2 - c)\langle\hat{z}^2\rangle$$
$$\frac{d}{d\tau}\langle\hat{p}_z^2\rangle = -2(\nu_z^2 - c)\langle\hat{z}\hat{p}_z\rangle \qquad (15)$$

where the coefficients a, b, c and d are given in terms of the second moments $\langle\hat{x}^2\rangle$, $\langle\hat{x}\hat{s}\rangle$, $\langle\hat{s}^2\rangle$ and $\langle\hat{z}^2\rangle$ via Eqs. (11) - (14). As a remark we note that the latter three of Eqs. (15) can be reduced to one second order differential equation for the vertical RMS-envelope $z_m = \langle\hat{z}^2\rangle^{1/2}$. This equation contains the vertical RMS-emittance ϵ_z (see for example Ref. (2)). In our case, this RMS-emittance is an integral of motion. This is due to our linear approximation of the space charge forces (see also Ref. (3) where RMS emittance-change is related to the change of non-linear field energy). A special solution of Eqs. (15) is obtained if one considers bunches with rotational symmetry around their vertical axis. For that case the equations for the horizontal moments can be reduced to one second order differential equation for the horizontal envelope of the bunch (9). Another special solution of Eqs. (15) is the stationary solution. However, this solution is only physically realistic for the rotationally symmetric bunch (9).

5. Integrals of motion of the moment equations

As already stated, the vertical RMS-emittance is a constant, due to the linear forces assumed. As a result of the transverse-longitudinal coupling in the unperturbed Hamiltonian, the horizontal and longitudinal emittances are not constant. However, the following "sum of emittances" defined by:

$$\epsilon^2 = \langle\hat{x}^2\rangle\langle\hat{p}_x^2\rangle - \langle\hat{x}\hat{p}_x\rangle^2 + \langle\hat{s}^2\rangle\langle\hat{p}_s^2\rangle +$$
$$- \langle\hat{s}\hat{p}_s\rangle^2 + 2(\langle\hat{x}\hat{s}\rangle\langle\hat{p}_x\hat{p}_s\rangle - \langle\hat{s}\hat{p}_x\rangle)$$

is a constant as can be verified with Eqs. (15). Another constant is the RMS-representation of the four-dimensional horizontal-longitudinal phase-space volume. For this we find the following expression:

$$\tau = 16\,\{[\langle\hat{x}^2\rangle\langle\hat{p}_x^2\rangle - \langle\hat{x}\hat{p}_x\rangle^2][\langle\hat{s}^2\rangle\langle\hat{p}_s^2\rangle - \langle\hat{s}\hat{p}_s\rangle^2] +$$
$$[\langle\hat{x}\hat{s}\rangle\langle\hat{p}_x\hat{p}_s\rangle - \langle\hat{x}\hat{p}_s\rangle\langle\hat{s}\hat{p}_x\rangle]^2 - [\langle\hat{x}^2\rangle\langle\hat{s}^2\rangle\langle\hat{p}_x\hat{p}_s\rangle^2 +$$
$$+ \langle\hat{p}_x^2\rangle\langle\hat{p}_s^2\rangle\langle\hat{x}\hat{s}\rangle^2 + \langle\hat{x}^2\rangle\langle\hat{p}_s^2\rangle\langle\hat{s}\hat{p}_x\rangle^2 + \langle\hat{s}^2\rangle\langle\hat{p}_x^2\rangle\langle\hat{x}\hat{p}_s\rangle^2] +$$
$$+ 2[\langle\hat{x}^2\rangle\langle\hat{s}\hat{p}_s\rangle\langle\hat{p}_x\hat{p}_s\rangle\langle\hat{s}\hat{p}_x\rangle + \langle\hat{s}^2\rangle\langle\hat{x}\hat{p}_x\rangle\langle\hat{p}_x\hat{p}_s\rangle\langle\hat{x}\hat{p}_s\rangle +$$
$$+ \langle\hat{p}_x^2\rangle\langle\hat{s}\hat{p}_s\rangle\langle\hat{x}\hat{s}\rangle\langle\hat{x}\hat{p}_s\rangle + \langle\hat{p}_s^2\rangle\langle\hat{x}\hat{p}_x\rangle\langle\hat{x}\hat{s}\rangle\langle\hat{s}\hat{p}_x\rangle] +$$
$$- 2[\langle\hat{x}\hat{p}_x\rangle\langle\hat{s}\hat{p}_s\rangle\langle\hat{x}\hat{s}\rangle\langle\hat{p}_x\hat{p}_s\rangle + \langle\hat{x}\hat{p}_x\rangle\langle\hat{s}\hat{p}_s\rangle\langle\hat{x}\hat{p}_s\rangle\langle\hat{s}\hat{p}_x\rangle]\}^{1/2}$$

We note that the quantities ϵ and τ are conserved for all linear canonical systems with two degrees of freedom and arbitrary coupling. Two other integrals of the smoothed moment equations are the (scaled) total angular canonical momentum $\bar{L}$ of the bunch defined as $\bar{L} = N(\langle\hat{s}\hat{p}_x\rangle - \langle\hat{x}\hat{p}_s\rangle)$, with N the total number of particles, and also the (scaled) total energy of the bunch $\bar{U} = \bar{T} + \bar{W}$ where $\bar{T}$ is the total kinetic energy ($\bar{T} = N\langle\hat{H}_o\rangle$ with $\hat{H}_o$ given in Eq. (5)) and $\bar{W}$ is the total space charge energy. For the ellipsoidal charge distribution we find (9):

$$\bar{W} = N(\langle\bar{x}\bar{E}_x\rangle + \langle\bar{s}\bar{E}_s\rangle + \langle\bar{z}\bar{E}_z\rangle) = \frac{3N}{2}\frac{I}{I_o}\frac{k}{A}\int_0^\infty \frac{d\mu}{(1+\mu)^{1/2}(\frac{B^2}{A^2}+\mu)^{1/2}(\frac{C^2}{A^2}+\mu)^{1/2}}$$

6. Conclusion

Within the approximations made, the numerical integration of the moment equations derived, gives the time evolution of the RMS-properties of the bunch under space charge conditions. A main advantage of the method is that the integration will ask for much less computer time than many-particle codes. As a first approximation, the assumptions made do not seem to be unreasonable. Therefore we expect that the model can give a useful contribution to the study of space charge effects in an AVF-cyclotron. Nevertheless, its possibilities and restrictions should be further evaluated by comparing the results with many-particle calculations.

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