A PARTICLE TRACKING METHOD FOR ACCELERATORS WITH SMALL RADIUS OF CURVATURE

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Abstract

We present a third-order numerical integration method applicable to particle tracking for accelerators with small radius of curvature. The present method considers the nonlinearity due to the small radius of curvature in equations of particle motion in a bending magnet and consists of explicit and symplectic transformations.

Chromaticity and stability limit of the betatron oscillation in an accelerator are evaluated and the results are compared with those obtained by the conventional linear matrix method for bending magnets.

Introduction

For accelerators with small radius of curvature, the nonlinearity in the equations of particle motion in a bending magnet is expected to be important for beam dynamics. For example, it has been pointed out that chromaticity calculation methods based on the linear transfer matrix approximation for a bending magnet are not correct for small accelerators[1]. Similarly, it is expected that this nonlinearity should also be considered in particle tracking for analysis of the dynamic aperture. In particle tracking, usually radiation is neglected and the particle motion is described by Hamilton's equation of motion. Therefore, a particle tracking method is required to satisfy the symplectic condition[2] for the Hamiltonian system. Numerical integration is considered to be the most straightforward way for particle tracking. Ruth[3] has developed a numerical integration method satisfying the symplectic condition. However, there is not a numerical integration method which is applicable to the Hamilton's equation including the nonlinearity due to the small radius of curvature and satisfies the symplectic condition.

In this paper, we present a numerical integration method considering the above nonlinearity and satisfying the symplectic condition. The present method has an accuracy up to the third order of the integration step width in Taylor's expansion. As a numerical example, the present method is applied to particle tracking for an accelerator with small radius of curvature. The results are compared with those obtained by the conventional linear matrix method for a bending magnet.

A Particle Tracking Method

The motion of a charged particle in a bending magnet can be described in curvilinear coordinates shown in Fig.1 by the following Hamiltonian H :

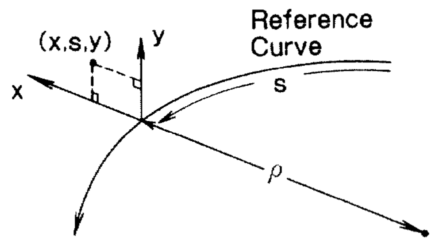

Fig.1 Curvilinear coordinate system

$$H=-\frac{q}{P}As-\left(1+\frac{x}{\rho}\right)\sqrt{1-P^2x-P^2y}\ , \tag{1}$$

where q is the charge; ρ, the radius of curvature; P, the particle momentum; and Px and Py are momenta along the x and y directions, normalized by the particle momentum P. A_s is the vector potential of the magnetic field and expressed as follows:

$$As=-Bo\rho\left[\frac{x}{\rho}+\left(\frac{1}{\rho^2}-K\right)\frac{x^2}{2}+\frac{Ky^2}{2}+\frac{S}{6}(x^3-3xy^2)+\cdots\right], \tag{2}$$

where B_o, K, and S are the dipole, quadrupole and sextupole field strengths. Since P_x and $P_y \ll 1$, the Hamiltonian H can be approximated as

$$H=-\frac{q}{P}As-\left(1+\frac{x}{\rho}\right)+\frac{P^2x+P^2y}{2}+\frac{x(P^2x+P^2y)}{2\rho}\ . \tag{3}$$

In the case of $\rho \gg x$, the last kinematic term on the right hand side, $x(P_x^2+P_y^2)/2\rho$ can be neglected. In this case, if the magnetic field is a dipole, that is, $S=K=0$, particle motion can be described analytically. In the case of a small radius of curvature, however, the last kinematic term on the right hand side of Eq.(3) cannot be neglected. Then, the equations of motion become nonlinear and are expressed as follows:

$$\frac{dx}{ds}=\frac{\partial H}{\partial Px}=\left(1+\frac{x}{\rho}\right)Px\ , \tag{4}$$

$$\frac{dPx}{ds}=-\frac{\partial H}{\partial x}=\frac{q}{P}\frac{\partial As}{\partial x}+\frac{1}{\rho}-\frac{P^2x+P^2y}{2\rho}\ , \tag{5}$$

$$\frac{dy}{ds}=\frac{\partial H}{\partial Py}=\left(1+\frac{x}{\rho}\right)Py\ , \tag{6}$$

$$\frac{dPy}{ds}=-\frac{\partial H}{\partial y}=\frac{q}{P}\frac{\partial As}{\partial y}\ . \tag{7}$$

In order to solve these equations of motion, we developed a numerical integration method having an accuracy up to the third order of an integration step width in Taylor's expansion.

The present numerical integration method employs the following transformations.

$$x_{i+1}=x_i+\left(1+\frac{x_{i+1}}{\rho}\right)\left\{h\,a_i\,P_{xi}+\frac{h^2}{2\rho}\,(c_i\,P^2xi+d_i\,P^2yi)+\frac{h^3}{\rho^2}e_i\,Pxi\;P^2yi\right\}\ , \tag{8}$$

$$y_{i+1}=y_i+\left(1+\frac{x_{i+1}}{\rho}\right)\left\{h\,a_i\,P_{yi}+\frac{h^2}{\rho}\,d_i\,Pxi\;Pyi+\frac{h^3}{\rho^2}e_i\,P^2xi\;Pyi\right\}\ , \tag{9}$$

$$P_{xi+1}=P_{xi}+h\,b_i\,Fx(x_{i+1},y_{i+1})-\frac{h}{2\rho}a_i(P^2xi+P^2yi)$$
$$-\frac{h^2}{\rho^2}\left(\frac{c_i}{6}P^3xi+\frac{d_i}{2}PxiP^2yi\right)-\frac{h^3e_i}{\rho^3 2}P^2xiP^2yi\ , \tag{10}$$

$$P_{yi+i}=P_{yi}+h\,b_i\,Fy\,(x_{i+1},y_{i+1})\ , \tag{11}$$

where a_i, b_i, c_i, d_i and e_i are constants, and h is the step width of integration. Fx and Fy are functions defined as follows:

$$Fx=\frac{q}{P}\frac{\partial As}{\partial x}+\frac{1}{\rho}\ , \tag{12}$$ $$Fy=\frac{q}{P}\frac{\partial As}{\partial y}\ . \tag{13}$$

It is easily shown that the above transformations satisfy the symplectic condition[2] to conserve the canonical character for Hamilton's equations of motion. In the above transformations of Eq.(8)-(11), it is noteworthy that the new coordinates x_{i+1}, y_{i+1} and new momenta Px_{i+1}, Py_{i+1} can be explicitly calculated. Then, by successively substituting and expanding up to the third order of the step h for Eqs.(8)-(11), the coordinates x_4, y_4 and the momenta Px_4, Py_4 can be expressed by initial values of x_1, y_1, Px_1 and Py_1. The obtained formulae for new coordinates and new momenta should be idential with the Taylor's expansion up to the third order of h. Comparing each term of the expanded formulae and Taylor's expansions, fifty nine relationships, many of which are dependent on each other, are obtained. The relationships for the constants of a_i and b_i $(i=1,2,3)$ are obtained from the terms up to the third order of h as follows:

$$a_1+a_2+a_3=1\ ,\quad b_1+b_2+b_3=1\ ,$$
$$a_2b_1+a_3(b_1+b_2)=\frac{1}{2}\ ,\quad a_1a_2b_1+a_1a_3(b_1+b_2)+a_2a_3b_2=\frac{1}{6}\ , \tag{14}$$
$$a_1b_1+(a_1+a_2)b_2+b_3=\frac{1}{2}\ .$$

There are five equations for six unknowns. A set of constants of $a_1=7/24$, $a_2=3/4$, $a_3=-1/24$, $b_1=2/3$, $b_2=-2/3$, and $b_3=1$ are chosen, by referring to the relationships shown in Ref.[3]. Furthermore, setting both e_1 and e_2 as zero, the constants of c_i, d_i $(i=1,2,3)$ and e_3 are determined uniquely from the rest of the relationships, which are omitted here, as follows:

$$c_1=\frac{4165}{15744}\ ,\quad c_2=-\frac{27}{32}\ ,\quad c_3=-\frac{6215}{15744}$$

$$d_1=\frac{833}{28800}\ ,\quad d_2=-\frac{9}{32}\ ,\quad d_3=-\frac{2083}{28800}\ ,\quad e_3=-\frac{99127}{2361600}\ .$$

Using these constants, we obtain the third-order explicit integration scheme satisfying the symplectic condition.

Numerical Example

As a numerical example, the present method was applied to an accelerator, which consists of four superperiods of the FODO unit cell. The unit cell is shown schematically in Fig.2.

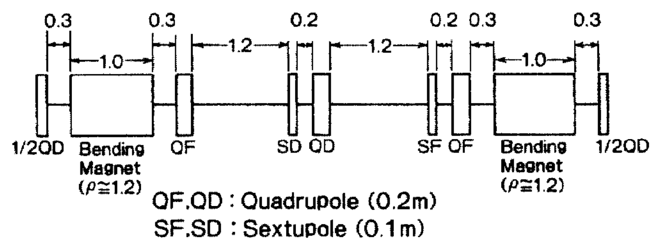

Fig.2 FODO unit cell (dimensions:m)

The radius of curvature ρ is about 1.2m and the magnetic field of the bending magnet is assumed to be purely dipole. Horizontal tune ν_x and vertical tune ν_y are 2.75 and 2.25. Field strengths normalized by $B_0\rho$ are -3.87 $1/m^2$ for the focuing quadrupole QF and 4.49 $1/m^2$ for the defocusing quadrupole QD. Momentum dependences of the horizontal and vertical tunes were obtained from the tracking results by the present and conventional linear matrix methods for bending magnets. In these methods, horizontal and vertical emittances were assumed to be 1 πmm·mrad. These results are shown in Fig.3. For comparison, results obtained by the semi-analytical formulae of Jäger and Möhl[4] for chromaticities which consider the effect due to the small radius of curvature are also shown in the figure. It is found the present results agree well with those of Jäger and Möhl. Two families of sextupoles SF and SD were used for chromaticity correction. The field strengths of the sextupoles SF and SD normalized by $B_0\rho$ are determined as 21.1 $1/m^3$ and 27.9 $1/m^3$, respectively, based on the results of the present and the Jäger-Möhl methods.

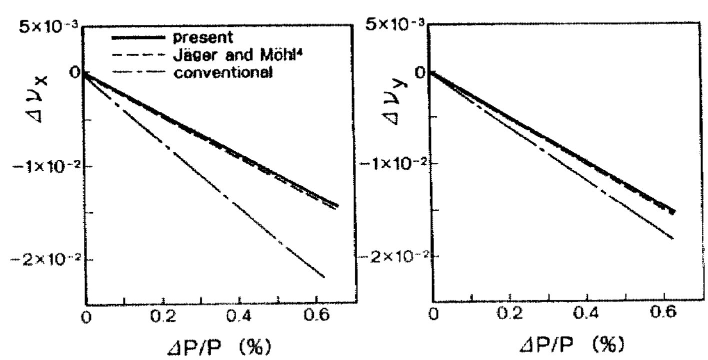

Fig.3 Momentum deviation ΔP/P vs. horizontal tune change $\Delta\nu_x$ or vertical tune change $\Delta\nu_y$

Stability limits of the betatron oscillation in the lattice shown in Fig.2 were analyzed by the present and conventional methods. Separatrices at a focusng quadrupole position obtained by tracking in one degree of freedom are shown in Figs.4 and 5 for each tracking method. A difference is seen between the stability limits obtained by the present and conventional linear matrix methods. It seems that the stability limits are determined by the third-order resonance for both cases. Then, the frequency spectrum of the betatron oscillation was analyzed for the case of the initial values of x=27mm and Px=0. While the main frequency component by the conventional method is about 2.74, the present method results in the main frequency component of 2.69, which is nearer the third-order resonance. Accordingly, the larger tune shift in the present method, which is due to the effect of the nonlinear kinematic term, caused the smaller stability limit. These results show that the nonlinear kinematic term should be considered in particle tracking for small accelerators.

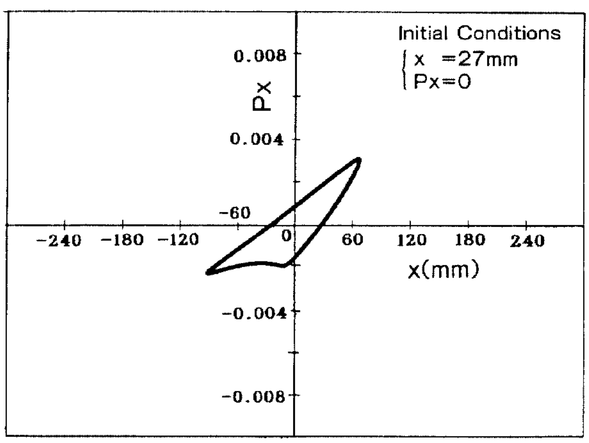

Fig.4 Phase space trajectory by the present method

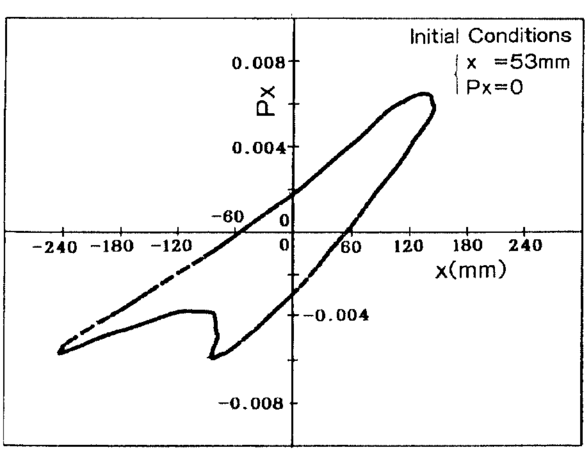

Fig.5 Phase space trajectory by the conventional method

Conclusion

A third-order numerical intergration method consisting of symplectic transformations was presented for application to particle tracking in accelerators with a small radius of curvature. In the present method, the nonlinearity in the equations of motion due to the small radius of curvature was considered. Chromaticities and stability limits of the betatron motion of an accelerator were evaluated by the present and linear matrix methods and it was shown that the nonlinearity due to the small radius of curvature should be considered in particle tracking.

References

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