HIGH-FREQUENCY LIMIT OF THE LONGITUDINAL IMPEDANCE OF AN ARRAY OF CAVITIES

S. A. HEIFETS
Continuous Electron Beam Accelerator Facility (CEBAF)
12000 Jefferson Avenue, Newport News, VA 23606
and
S. A. KHEIFETS
Stanford Linear Accelerator Center (SLAC)
2575 Sand Hill Road, Menlo Park, CA 94025

Abstract

The longitudinal impedance of a linear array of cylindrically symmetric cavities connected by side pipes is estimated in the high-frequency limit. The expression for the impedance is obtained for an arbitrary number of the cavities. The impedance per cell decreases with frequency ω, as $\omega^{-1/2}$ for a small number of cells, and as $\omega^{-3/2}$ for the infinite periodic structure. The parameter is given which governs the transition from one regime to another.

The next generation of linear colliders, FEL drivers, and synchrotron light sources will use bunches with small bunch lengths. Evaluations of the stability of the particle motion and of the current limitations in all such devices require an accurate estimate of the impedances for the high-frequency region.

There is a certain discrepancy at the present time in the estimates of the longitudinal impedance obtained for different models. The optical resonator model,[1] which is applied to an infinite periodic set of thin discs, predicts a decrease of the longitudinal impedance with frequency as $\omega^{-3/2}$, and some numerical calculations are consistent with this result.[2] On the other hand, the analytical evaluations of the longitudinal impedance for a single cavity[4],[5] give quite a different behavior; the impedance goes down as $\omega^{-1/2}$. This dependence for a single cavity was also obtained in a simple diffraction model.[3],[6],[7] As we show in the present paper the different behavior of the high-frequency impedance is related to the differences of the models used.

Here we present an analytical evaluation of the real part of the longitudinal impedance for any number M of identical cylindrically symmetric cells. Each cell consists of a cavity with side pipes. The length of a cavity is g, and the radii of the cavity and the pipe are b and a, respectively. The total length of a cell is L. At both the entrance and the exit the system of cells is connected to semi-infinite pipes of the same radius a. The frequencies ω under consideration are well above the cut-off frequency of the pipe, but at the same time are small in comparison to the particle Lorentz factor γ: $1 \ll \omega a/c \ll \gamma$. The number of cavities M is considered to be a variable parameter.

For cylindrically symmetric (monopole) modes, Fourier harmonics of the electric field which are generated by a particle with the charge e and the velocity v moving along the axis of the system, can be written as a sum of the field of a particle in a straight pipe of the radius a and the radiation field E^{rad}, produced due to the presence of the cavities. The radiation field E^{rad} satisfies the homogeneous wave equation and has to be finite at $r = 0$. It can be represented as a superposition of cylindrical eigenfunctions with unknown coefficients $A(q)$, for example:

$$E_z^{rad}(r,z) = \frac{e}{\pi v}\int_{-\infty}^{\infty} dq A(q) J_0(\chi_q r/a) e^{iqz} \tag{1}$$

where $\chi_q = a\sqrt{k^2 - q^2 + i\epsilon}$. An infinitely small imaginary part ϵ is added to comply with the radiation condition.

We look for the solution in the form

$$A(q) = \sum_{N=0}^{M-1}\sum_{n=0}^{\infty} \frac{V_n(q)}{J_0(\chi_q)} B_n^N e^{i(k-q)NL} . \tag{2}$$

Boundary condition $E_z^{rad}(a,z) = 0$ for $g < z - NL < L$ and matching conditions for the radial and tangential components of the field in the pipe and in the N-th cavity at $r = a$, $0 \le z - NL < g$ give the following integral equation for B_n^N:

$$B_n^N = \frac{a}{\pi g} C_n(k)\left\{\frac{i}{ka^2}V_n^*(k) + \sum_{N'=0}^{M-1}\sum_m \Gamma_{nm}^{N-N'} B_m^{N'}\right\} \tag{3}$$

where $N = 0, 1, ..., M-1$. Here the following notations are introduced:

$$C_n(k) = \sqrt{k^2 - \lambda_n^2}\tan\left((b-a)\sqrt{k^2-\lambda_n^2}\right); \qquad \lambda_n = \frac{n\pi}{g} \tag{4}$$

$$V_n(q) = i\frac{q}{\lambda_n}U_n^*(q) = \int_0^g dz e^{-iqz}\cos(\lambda_n z) \tag{5}$$

The matrix elements Γ_{nm} are :

$$\Gamma_{nm}^{N-N'} = \sum_l \frac{2\pi i}{u_l a^2}\begin{cases} V_n^*(u_l)V_m(u_l)e^{iL(u_l-k)(N-N')} \\ V_m^*(u_l)V_n(u_l)e^{-iL(u_l+k)(N-N')} \end{cases}, \tag{6}$$

for $N \ge N$ and $N < N$ respectively, where $u_l = \sqrt{k^2 - (\nu_l/a)^2}$ and ν_l are roots of the equation $J_0(\nu) = 0$. In the sum of Eq. (6) all terms with $\nu_l > ka$ are exponentially small. Hence, the summation over l may be truncated at $\nu_l = ka$.

The longitudinal impedance in terms of the coefficients B_n^N is

$$Z(k) = -Z_0 \sum_{N=0}^{M-1}\sum_{n=0}^{\infty} V_n(k) B_n^N(k) . \tag{7}$$

So far, Eq. (3) is the exact set of equations defining the radiation of an ultrarelativistic particle.

In the *zero*-th order approximation we neglect the second term in the brackets in Eq. (3):

$$B_n^N = \frac{iC_n(k)}{\pi gka} V_n^*(k) \tag{8}$$

Notice that in the zero-th approximation the impedance per cell does not depend on the number of cells in the array.

The real part of the impedance is given by the sum of δ-functional terms $\delta(k-k_{nl})$ where the resonance frequencies are defined by equation $C_n^{-1}(k_{nl}) = 0$.

The main contribution to the impedance comes with a good accuracy from eigenmodes with the eigennumbers

$$n = n_0(k) = \left[\frac{kg}{\pi}\right] \qquad \text{and} \qquad 0 \le l \le l_{max}\,. \tag{9}$$

This result has a very simple physical meaning: only these eigenmodes efficiently interact with a relativistic particle traversing a cavity. The impedance averaged over the interval of frequencies $\Delta k \approx \pi/2g$ differs from Lawson's estimate[3]

$$< \mathrm{Re}\left(\frac{Z}{M}\right) >= \frac{Z_0}{2\pi}\sqrt{\frac{g}{\pi a}}\frac{1}{\sqrt{ka}} \tag{10}$$

only by a numeric factor $\pi/3$. Numerical calculations confirm that this result is independent of the choice of the size of the interval Δk.

The zero-th approximation for a single cavity can be improved by taking into account that the main contribution to the sum in Eq. (3) is given by the diagonal term $m = n$. All the other terms give only small corrections which can be taken into account by the method of iterations. In this *diagonal approximation*[5] the impedance can be represented as a sum over the Breit-Wigner terms. The resonance frequencies are now given by the condition

$$\mathrm{Re}\, y(k) = 0, \qquad y(k) \equiv \frac{\pi g}{a} C_n^{-1} - \Gamma_{nn}^0$$

and finite resonance widths are defined by Im Γ_{nn}^0. ReZ is not singular, as it was in the zero-th approximation, although it may have rather sharp peaks. This is the main qualitative feature of the diagonal approximation for a single cavity. The result for ReZ/M is the same as in Eq. (10). The diagonal approximation allows us to estimate corrections, (given by the next iterations) and proves that in high-frequency limits they are small.[5]

Consider now a structure consisting of M cells. The interference of the waves generated in different cells is crucial in the evaluation of the impedance for the multi-cell structure, and must be taken into account. We describe the interaction of a particle with each cell in the same way as we did above for a single cavity. Therefore, we consider Eq. (3) in the diagonal approximation for the lower indices, retaining only terms $m = n = n_o$, but keeping the summation over the upper indices N'.

Furthermore, all terms in Eq. (3) with $N < N'$ contain factors which oscillate with the sum frequencies $u_l + k \sim 2k$. After averaging over the frequency interval they would only give a negligibly small contribution. Therefore, we may assume that $\Gamma_{nn}^{N-N'} = 0$ for $N < N'$ and retain only terms with $N > N'$ which oscillate with the small difference frequencies $(u_l - k)$. Equation (3) takes the form

$$B_n^N y(k) = \frac{iV_n^*}{ka^2} + \sum_{N'=0}^{N-1} \Gamma_{nn}^{N-N'} B_n^{N'}\,. \tag{11}$$

Equation (11) shows the recurrence relations between coefficients B_n^N. It can also be solved by the discrete Laplace transformation. The impedance (Eq. 7) of an array with an arbitrary number of cells M is given by the following expression:

$$Z(k) = -\sum_{n=0}^{\infty} \frac{Z_0|V_n(k)|^2}{4\pi ka^2}\cdot \int_{-i\pi+\sigma}^{i\pi+\sigma} ds \frac{e^{Ms}-1}{(y(k)-\Gamma_n(s,k))(\cosh s - 1)} \tag{12}$$

Here

$$\Gamma_n(s,k) = \sum_{l=0}^{\infty} \frac{2\pi i|V_n(u_l)|^2}{u_l a^2}\frac{1}{e^{iL(k-u_l)+s}-1} \tag{13}$$

and

$$u_l = \sqrt{k^2-(\nu_l/a)^2+i\epsilon}\,. \tag{14}$$

The contour of the integration can be closed by two parallel lines $s = -i\pi+\sigma$ and $s = +i\pi+\sigma, -\infty < \sigma < 0$. The integral is then equal to the sum of the residues at the root of the equation $\cosh s = 1$ and at the roots of the equation

$$y(k) = \Gamma_n(s,k). \tag{15}$$

For an array with small M it is convenient to rewrite Eq. (12) introducing a new variable $t = e^{-s}$. This inverts the infinite point to zero and transforms the contour of integration to a circle with a radius $|t| < 1$. The only singularity inside the contour is at the point $t = 0$. The integral is given by a finite double sum. If the following condition is fulfilled:

$$\frac{ML}{2k}\left(\frac{\pi}{a}\right)^2 \ll 1\,, \tag{16}$$

the average impedance per cell is now given as an expansion:

$$< \mathrm{Re}\left(\frac{Z}{M}\right) >= \frac{Z_0}{2\pi}\sqrt{\frac{g}{\pi ka^2}}\left[1+\frac{2}{5k(b-a)}\sqrt{\frac{\pi M}{kL}}+\dots\right] \tag{17}$$

in a parameter $\tau \sim M^{1/2}/(ka)^{3/2}$. In this case the real part of the impedance per cell is the same as that for a single cavity. For large M, expansion (Eq. 17) is not applicable. This case requires the detail analysis of Eq. (15). We can show that the average impedance for an arbitrary M may be written as

$$< \mathrm{Re}\left(\frac{Z}{M}\right) >= \frac{2Z_0}{(ka)^{3/2}}\left(\frac{2L}{\pi a}\right)^2\sqrt{\frac{\pi a}{g}}\Phi(k,M) \tag{18}$$

where

$$\Phi(k,M) = k\pi g\left(\frac{a}{4L}\right)^2 \int_{-\pi}^{\pi} \frac{dt}{2\pi}\frac{F(t,M)}{\xi^2}\left(1-\frac{\arctan\xi}{\xi}\right). \tag{19}$$

Here

$$\xi(t,k) = \frac{a}{2L}\sqrt{k\pi g}\frac{J_1(r)}{rJ_0(r)}; \qquad r = \sqrt{-\frac{2ka^2}{L}t}$$

and

$$F(t,M) = \frac{\sin^2(Mt/2)}{M\sin^2(t/2)} \tag{20}$$

are introduced. For an infinite periodic array of cavities

$$\lim_{M\to\infty}F(t,M) = 2\pi\delta(t)\ . \tag{21}$$

For function $\Phi(k,M)$ we obtain in this case

$$\Phi(k,\infty) = 1 - \frac{2\pi L}{a\sqrt{\pi k g}} \tag{22}$$

and the average real part of the impedance per one cell is:

$$< \mathrm{Re}\left(\frac{Z}{M}\right) >= \frac{2Z_0}{(ka)^{3/2}}\left(\frac{2L}{\pi a}\right)^2\sqrt{\frac{\pi a}{g}} + O(k^{-2})\ . \tag{23}$$

It decreases with frequency as $(ka)^{-3/2}$.

For finite $M >> ka$, the estimate of the factor Φ gives

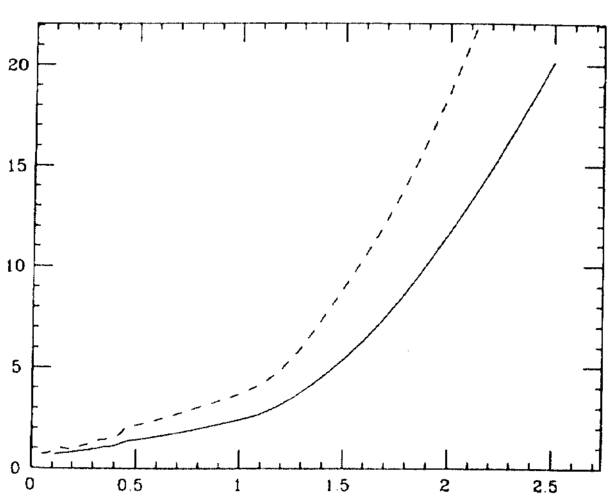
$$\Phi = 1 + \frac{1}{48M}\left(\frac{ka^2}{L}\right)^{3/2}\sqrt{\frac{\pi g}{L}}\ . \tag{24}$$

For $M \to \infty$ $\Phi = 1$, as is shown above, the real part of the average impedance decreases with frequency as $(ka)^{-3/2}$. The same is true if the second term in Eq. (24) is small. For a large given M the transition from the regime $(ka)^{-1/2}$ (Eq. 17) to the regime $(ka)^{-3/2}$ (Eq. 23) takes place in the range

$$\frac{ka^2}{L} \ll M \ll \left(\frac{ka^2}{L}\right)^{3/2}\sqrt{\frac{\pi g}{L}}\ . \tag{25}$$

The transition from one regime to another is illustrated in Fig. 1. The curves represent function Φ versus ka^2/ML for different values M and are obtained by numerical integration of Eq. (19). The data are in agreement with the analytical estimates given above.

transition to a periodic array



Factor Φ Defined By Eq. (19) vs. (ka^2/ML); M= 1500(s),3500(d)

Conclusion

Our result is based on the solution of the exact system (Eq. 3), derived from the Maxwell equations with the appropriate boundary conditions. To obtain the solution we have done two approximations. First, we have shown that there are only a few modes in the cavities which substantially interact with a relativistic particle in the high-frequency limit. This approximation is independent of the number of cavities in the array and for a single cavity gives the correct result within a factor of order of one. Second, we neglected the interaction of a particle with waves traveling in the opposite direction. This reduces the infinite set in Eq. (3) to the recurrence equations in the form of Eq. (11). They are solved explicitly with the result given by Eq. (12). The interference and the phase difference of the waves generated in different cells is taken into account. Since we are interested only in the impedance averaged over frequency there is no need to calculate the exact frequencies of the eigenmodes for the array. The explicit form of the averaged impedance is given in two extreme cases of small and large numbers of cells. The frequency dependence of the real part of the impedance has a direct implication on the design of a short bunch accelerator; had the asymptotic decrease of the longitudinal impedance followed the law $k^{-1/2}$, the main contribution to the total energy loss would be given by the high-frequency tail of the impedance and the total energy loss would depend on the longitudinal rms of the bunch σ as $\sigma^{-1/2}$.

The appropriate parameters of two accelerators — Stanford Linear Collider (SLC) and TeV Linear Collider (TLC) — show that for both designs the impedance falls off as $k^{-3/2}$. However, the parameter M could be smaller than the total number of cavities for different reasons (sectioning of the accelerator, differences in the dimensions, production errors, etc.) which could change the impedance behaviour for typical frequency $k \sim 1/\sigma$, and the total energy loss may increase.

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