ANOTHER LOOK AT COHERENT LONGITUDINAL INSTABILITIES OF BUNCHED BEAMS

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Abstract

Existing theories predict that, the higher the mode number is of a longitudinal bunched beam instability, the higher is likely to be the frequency (or range of frequencies) of the instability excitation fields. In this paper a different result is found with all modes having a similar upper frequency limit for the fields but with the odd and even order modes having a dissimilar range for the possible frequencies. The relevance for uncoupled bunch, coupled bunch and mode coupling instabilities is discussed.

Introduction

A bunch train with a stationary distribution of particles in the bunches has a spectrum that extends up to a frequency approximately equal to the inverse of the bunch duration. The bunch harmonics are the only carrier frequencies available to sustain a longitudinal bunched beam instability. Yet the general theory, as developed by Sacherer [1] and others, predicts that the instability excitation frequencies may be very much higher than the bunch frequencies. Form factors are derived for the different instability modes which indicate that the higher the mode number, the higher is likely to be the excitation frequency. If this prediction of the general theory is correct, how is the beam able to convey information at excitation frequencies higher than the bunch frequencies?

The general theory is developed from the one dimensional Vlasov equation, with the use of polar co-ordinates (r,θ) in the longitudinal phase space $(\phi,\dot{\phi})$. A two dimensional density in this phase space, $\Psi(r,\theta)$, is assumed to be of the form:

$$\Psi_o(r) + \Sigma_m R_m(r) e^{im\theta} e^{ip(m)t}$$

where m is the instability mode number.

The simplest case to consider is the dipole mode of instability (m=1) for uncoupled bunch motion (n=0). Then, it is possible to re-arrange the Vlasov equation and solve directly for the radial function, $R_1(r)$, leaving a phase equation for coherent dipole motion to be solved. This is the equation originally solved by K W Robinson [2] for a beam interacting with the fundamental resonance of the accelerating cavities, in the absence of any feedback systems; the solution is a fourth order equation which may have unstable roots in two different frequency regions. It is of interest, for this specific case, to make a detailed comparison between the Robinson and the general theory, as the latter leads to a first and not a fourth order equation.

For comparison, it is convenient to consider three separate expressions, the coherent phase equation, the cavity transfer function for modulations of drive current and an expression for the modulation components of beam current. In each case, Robinson uses a different relationship than that developed in the general theory. Firstly, for the coherent phase equation, there is one perturbation due to the phase modulation and one due to the amplitude modulation of the accelerating voltage; Robinson uses both, whereas the general theory neglects the phase modulation term. This is an important omission, for coherent dipole motion involves the both terms (while the odd and even higher order modes involve only the former and latter respectively). Secondly, for the beam cavity interaction, Robinson correctly uses the equivalent of a complex transfer function, whereas the general theory evaluates the products of the complex cavity impedance and beam current component at each beam frequency, with a correction term for the cavity bandwidth. Thirdly, for the beam current modulations, a major difference emerges between the two theories. Robinson considers dipole motion as phase modulation of the fundamental beam current component. The general theory uses a Fourier expansion of the beam current modulation in terms of the longitudinal phase space parameter Φ. Such an expansion is invalid, however, for though the unperturbed line charge density is periodic in ϕ, the modulation is not. Thus, the form factors of the general theory are developed from an incorrect premise.

A modified theory is now developed, first extending the Robinson theory to include parasitic mode excitation of the dipole mode, and then to the case of coupled bunch dipole motion. Also considered is the special case of RF feedback for the main accelerating system. Higher mode motion is analysed from the Vlasov equation, separating the radial mode functions, $R_m(r)$, by analogy with the method for the dipole mode. The theory is simplified by the use of a dual complex number representation for modulated signals, with one complex number to represent the carrier and a second for the modulation components.

Revised form factors are developed and their relevance for mode coupling instabilities is discussed.

Uncoupled Dipole Mode Motion

The Sacherer form of the Vlasov equation for the dipole mode may be rearranged and approximated to the following form:

$$(s^2 + \Omega^2)\, R_1(r)\, \bar{R}\left(\frac{e^{i\theta}}{2i}\right) e^{ipt} = -\sin\theta \frac{d\Psi_o}{dr} (s^2 + \Omega^2)\phi$$

where (r,θ) are polar coordinates in $(\phi,\dot{\phi}/\Omega)$ space,
Ψ is the 2-D stationary bunch density,
$R_1^o(r)$ is the radial function for the m = 1 mode,
Ω is 2π x synchrotron oscillation frequency,
$\bar{R}$ is the real part of the argument $(\exp i\theta)/2i$,
ϕ is the coherent phase motion ($\hat{\phi}\exp(ipt)$) and
$s\ (\equiv d/dt) = ip$, with p the complex frequency.

Hence, $R_1(r) = -2\,\hat{\phi}\, d\Psi_o/dr$

and it remains to solve for p from the coherent phase equation (1), the complex cavity transfer function (2) and the beam current modulation (3):

$$(s^2 + \Omega^2)\,\phi = s^2(\Delta\phi_v) - \Omega^2 \sin\phi_s\,(v/V) \qquad \ldots(1)$$

$$v/V + j\Delta\phi_v = G(jw, ip)\,\Delta I_b/V \qquad \ldots(2)$$

$$\Delta I_b = I_b\,(\Delta\phi_v - \phi)\, e^{-j\phi_s} \qquad \ldots(3)$$

Here, V is the cavity voltage,
 v is the amplitude modulation of V,
 $\Delta\varphi_v$ is the phase modulation of V,
 ΔI_b is the beam current modulation and
 $G(jw, ip)$ is the dual complex number representation of the cavity transfer function.

For the phase modulation of the beam current, a linear approximation is assumed for the usual Bessel function expansion. The reference phase is that of V, and I_b is the amplitude of the relevant harmonic of the unperturbed beam current. Input modulation is represented by exp(jwt) exp(ipt), with w and p the angular frequencies of the carrier and modulation respectively. In this notation, a transfer function response is represented by:

$$\bar{R}_j\ (\bar{R}_i\ (G(jw,ip)\ e^{ipt})\ e^{jwt})\ /\ (\cos wt \cos pt)$$

where $\bar{R}_i$ is the real part of the complex argument with j assumed constant and $\bar{R}_j$ is the real part of the subsequent argument in which j is assumed complex. Typical G(jw,ip) functions for $\Delta V/\Delta I$ are:

$(jw + ip)\ L$ for an inductance L,

$((jw + ip)\ C)^{-1}$ for a capacitance C.

Such transfer functions are the composite response for both the upper and lower sidebands of the modulation. Alternatively, the response may be found for individual sidebands; for example, the responses for a cavity of loaded shunt resistance, R, loaded quality factor, Q, and angular resonant frequency, w_o, are:

$$R(1 \mp ij)\ /\ 2\ (1 + (2Q/w_o)(j\Delta w + ip))$$

where $\Delta w = w - w_o$ and the - and + signs refer to the upper and lower sidebands respectively. The terms in (ij) are the quadrature terms associated with single sideband modulation. There is cancellation of the (ij) terms for the composite response.

Sometimes it is necessary to consider the single-pass transient cavity response in addition to the multi-turn response. This is the case for very large rings where the revolution time is of the same order as the cavity time constant. To obtain such a single-pass response, it is necessary to know the detailed form of the beam cavity interaction or to assume an appropriate model for the cavity.

In the special case where RF feedback is used in the final stages of the cavity amplifier systems, there is a modified form of G(jw,ip) for the composite cavity response:

$$R(1 + A \exp(-(j\Delta w + ip)T) + (2Q/w_o)(j\Delta w + ip))^{-1}$$

Here A and T ($= 2\pi\ k/w_o$) are the open loop feedback gain and delay respectively, with k an integer. This form of G(jw, ip) is valid only if the single-pass response may be neglected; then equations (1), (2) and (3) may be combined to give a fourth order equation:

$$s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_o = 0$$

with $b_o = \Omega^2(C^2 + \Delta w^2 + CD\Delta w/((1 + A)\cos\phi_s)$

$b_1 = \Omega^2 b_3 = 2\ \Omega^2\ C$

$b_2 = (C^2 + \Delta w^2 + \Omega^2)$

$C = (1 + A)/(2Q/w_o - AT)$

$D = RI_b/V$

For the case of no RF feedback, A = T = 0 and the fourth order equation reduces to that of Robinson. For the case of RF feedback, there are three possibilities for instability. Two conditions are as for Robinson, viz instability when the cavity is detuned in the direction opposite to that required for beam loading compensation, and instability under reactive compensation above certain ratios of beam to cavity power. For no RF feedback, the ratio is unity; for RF feedback, the ratio may be much higher. The third condition for instability occurs if the feedback gain, A, is increased above the value $Q/k\pi$. The equation becomes of higher order when low frequency loops are added for control of cavity field amplitude, tuning, beam radial position and coherent phase motion. Then, there is coupling between the loops due to the beam loading but it is minimised by the use of the RF feedback.

The coherent phase motion is further modified if there is dipole mode excitation of both the main and parasitic cavity resonances. From equation (3), it may be seen that the beam current modulation is proportional to the amplitude of the associated beam current harmonic. Parasitic mode excitation may thus occur at any frequency within the unperturbed bunch spectrum, which extends up to the inverse of the bunch duration. This is in contrast to the prediction of the general theory, where the form factor for the dipole mode is shown to be finite up to frequencies twice as high. The parasitic resonance must be close to a multiple of the main RF frequency to influence the uncoupled dipole mode motion. The more general case is when the parasitic resonance is close to a harmonic frequency of the ring but not to a multiple of the main RF frequency. In this case, coupled bunch dipole or higher mode motion may be excited.

Coupled Bunch Dipole Mode Motion

If there are M circulating bunches in a ring ($M \leqslant h$), there are M coupled bunch modes of oscillation for each value of m (m = 1 for dipole, m = 2 for quadrupole.... and h is the RF harmonic number).

Coupled bunch motion may be excited via a parasitic or the main cavity resonance. Parasitic excitation of coupled or uncoupled dipole mode motion may be analysed as follows. Consider the case of h equally populated bunches undergoing dipole motion at coupled bunch number n (= 0, 1, ...(h-1)), with n = 0 corresponding to the uncoupled case. If only one value of n is excited at the complex coherent frequency ($\cong \Omega$), it may be shown that the additonal angular frequencies that appear in the beam spectrum, for a revolution frequency $w_r/2\pi$ are:

$$\ell h w_r \pm p_1 \text{ or } p_2 \quad \text{with } \ell = 0, 1 \ .....$$

$$p_1 = p + nw_r \quad , \quad p_2 = p - nw_r$$

The frequencies p_1 and p_2 converge for n = 0. The parasitic resonance (at w_{op}) is excited most strongly by the sidebands linked to one particular value of ℓ. The excitation may be obtained, as previously, by a composite response G(jw, ip), but with $\Delta w = \ell h w_r - w_{op}$ and the modulation frequencies at p_1 or p_2. For the former (p_1) and the latter (p_2), the sideband pairs are, respectively:

$$(\ell h + n)w_r + p \text{ and } (\ell h - n)w_r - p,$$

$$(\ell h + n)w_r - p \text{ and } (\ell h - n)w_r + p.$$

The coherent phase motion is found from modified forms of equations (1), (2) and (3). In (1), $\sin\phi_s$ is approximated by $\sin \ell\phi_s$; in (2), the new Δw is used in $G(jw, ip)$ and p is replaced by p_1 or p_2; and in (3) the following form of ΔI_b is used:

$$\Delta I_b = 2j\ I_\ell\ e^{j\ell(3\pi/2\ -\ \phi_s)}\ J_1(\ell(\Delta\phi_v - \phi))$$

where I_ℓ is the amplitude of the ℓ-th bunch frequency harmonic in the unperturbed beam current. It may be written in terms of an integral involving $J_o(\ell r)$ and $\psi_o(r)$. Combining (1), (2) and (3) gives a fourth order equation in s, but now with complex coefficients, involving terms in i. It may be reduced to two simultaneous fourth order equations, with variables the real and complex parts of the coherent frequency.

The motion becomes more complex when the bunches are not equally populated or when there are missing bunches. In this case, each bunch motion is described by a fourth order equation in terms of the motion of each of the other bunches. Then, for M bunches, there are M coupled fourth order equations to be solved.

Coupled bunch feedback systems sometimes use just single sideband modulation for the feedback. Such feedback may be analysed by use of the single sideband transfer functions given in the previous section. Coupled bunch frequencies appear in the spectrum in pairs at nearly the same frequencies eg $(\ell h + n \pm p)\ w_r$. Since the transfer functions are different for the two frequencies, feedback to damp one mode may antidamp the other and vice-versa. Then it is necessary to transform appropriately the individual frequency components.

Odd Order Higher Modes

The appropriate Vlasov equation is as before but with Ω (on LHS) replaced by $m\Omega$, $R_1(r)$ by $R_m(r)$ and the argument $(\exp\ i\theta)/2i$ by $(\exp\ im\theta)/(m+1)i$, where $m = 3, 5, 7 \ldots$

Excitation currents and fields are of the form $\exp(j\ wt)\ \exp(ipt)$ with $wt = \phi$ and $p = p_1$ or p_2. Then the amplitude modulations of field correspond to $\sin(\ell r \cos\theta)$ terms and the phase modulations to $\cos(\ell r \cos\theta)$ terms. These may be expanded in terms of Bessel functions.

$$\sin(\ell r \cos\theta) = 2\Sigma_1^\infty (-1)^{n+1} J_{2n-1}(\ell r)\ \cos(2n-1)\theta$$

$$\cos(\ell r \cos\theta) = 2\Sigma_1^\infty (-1)^n J_n(\ell r)\ \cos 2n\theta + J_o(\ell r)$$

From the form of the Vlasov equation and the Bessel function expansions, it is seen that the odd-order higher modes may only be excited by the phase modulation field components. Mode $m = 3$ excitation is related to the sum $J_2(\ell r) + J_4(\ell r)$ and hence to $J_3(\ell r)$:

$$J_{m+1}(\ell r) + J_{m-1}(\ell r) = 2mJ_m(\ell r)/\ell r$$

Similarly, mode $m = 5$ is related to $J_5(\ell r)$ and the order of the mode turns out to be the same as the order of the associated Bessel function. By analogy with the dipole mode, the radial function may be separated:

$$R_m(r) = (-1)^{(m+1)/2}\ (2m(m+1)\ J_m(\ell r)\ /\ell r)\ \frac{d\psi_o}{dr}\ \phi$$

The coherent motion for the odd order modes is characterised by an oscillation in phase and energy of the centre of gravity of the bunches. This corresponds to phase modulation of beam current with an amplitude given by the ΔI_b formula of the last section and a complex mode frequency, $\cong m\Omega$. Since ΔI_b is proportional to I_ℓ and is independent of m, the possible range of excitation frequencies corresponds to the unperturbed bunch spectrum and is the same for all the odd order modes.

The mode equations may now be developed (using small angle approximations) in terms of the phase motions of the bunch centres:

$$(s^2 + m^2\Omega^2)\ ((1 + (2Q/w_o)ip_{1,2})^2 + ((2Q/w_o)\Delta w)^2)\phi =$$

$$\Omega^2\ H\ \phi\ ((1 + (2Q/w_o)ip_{1,2})\ \cos\alpha + (2Q/w_o)\Delta w\ \sin\alpha)$$

with $H = m^2\ D\ \ell\ I_\ell/I_1$ and $\alpha = \ell(3\pi/2 - \phi_s)$.

Even Order Higher Modes

The Vlasov equation is unchanged but now the excitation is due to the amplitude modulation of the fields. Again the order of the mode is the same as the associated Bessel function excitation term.

Even order modes are characterised by no coherent motion of the bunch centre but by an oscillation of the bunch phase extent, ϕ_b. This corresponds to amplitude modulation of the bunch current harmonics at the complex mode frequency, with amplitudes:

$$\Delta I_\ell = (\partial I_\ell/\partial\phi_b)\ e^{j\ell(3\pi/2\ -\phi_s)}\Delta\phi_b$$

The derivative of I_ℓ with respect to ϕ_b now determines the excitation spectrum. The upper frequency limit to the spectrum is as for I_ℓ but the lower limit is different; eg for short bunches, the derivative is zero for the lower harmonics. All the even order modes may be excited over this reduced frequency range, and it is adequate for both even and odd modes to damp parasitic cavity resonances over the frequency range of the I_ℓ spectrum.

The radial functions for the even order modes are:

$$R_m = (-1)^{m/2}\ (2m(m+1)\ J_m(\ell r)/\ell r)\ \frac{d\Psi_o}{dr}\ \Delta\phi_b$$

and the mode equations are identical to those for the odd order modes, but with ϕ replaced by $\Delta\phi_b$ and:

$$H = -\ D\ (\partial I_\ell/\partial\phi_b)/I_1\ \cos\phi_s$$

Mode Coupling

The principle of mode coupling is as described in [1] but the detailed mechanism is different here due to the higher order of the equations and the different form factors. The form factors, H, are proportional to I_ℓ for the odd and to $(\partial I_\ell/\partial\phi_b)$ for the even order modes.

Coupling occurs when mode frequencies approach one another. Phase modulation of beam current excites both phase and amplitude modulation of V, the former driving odd and the latter even order modes. Amplitude modulation of I_ℓ then complements the coupling.

Mode frequencies may cross because of the different range of excitation spectra for the odd and even modes. The extended spectrum for the odd modes increases the inductive contributions to the frequency shifts. Certain combinations of impedances may lead to mode coupling both below and above transition energy, in contrast to earlier findings.

References

[1] F J Sacherer, Bunch Lengthening and Microwave Instability, CERN/PS/BR 77-6 (1977).

[2] K W Robinson, Stability of Beam in Radio Frequency System, CEAL-1010, 27 February 1964.