THE RENORMALIZED THEORY OF BEAM-BEAM INTERACTION*

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Abstract

A new approach to calculate analytically the particle distribution in presence of beam-beam interaction and synchrotron radiation effects for an electron-positron colliding beam storage ring is presented. The method is based on correct calculation of the Green's function which includes the full effect of the beam-beam force on the distortion of particle orbits, borrowing the renormalization technique of quantum field theory. By this way, the theory is applicable to any level of beam-beam interaction, no matter whether chaos ensues in phase space or not. This paper is devoted mostly to verification of the theory by comparison with the results of computer simulations. Fairly good agreements are obtained.

Introduction

The beam-beam interaction in a colliding beam storage ring has been extensively studied ever since the colliding beam rings were invented. Most of the works have been done in terms of the Hamiltonian analysis of single particle dynamics associated with the theory of nonlinear resonances excited by the beam-beam force. Some criterions (e.g., Chirikov, Green,...) are proposed as methods for estimating the onset of chaotic behavior of particle orbits. Despite its widely acknowledged success, the Hamiltonian analysis has the following drawbacks due to its nature of approach: (1) It breaks down when the particle motion becomes chaotic (unpredictable); (2) It does not explain the emittance growth or the particle loss which have been observed in real machines, since it says nothing about the beam population in resonances or chaotic regions in phase space; (3) It can be applied only to proton machines where the effects of synchrotron radiation are negligible. In order to overcome these drawbacks of Hamiltonian analysis and to have a more direct tool to estimate the emittance blow-up in an electron-positron collider, a few attempts [1,2,3] have been made to formulate a theory for analytical calculation of the particle distribution in presence of beam-beam interaction and the effects of synchrotron radiation. They use the Fokker-Planck equation for the evolution of the particle distribution. In spite of their restriction to the one-dimensional strong beam-weak beam interaction picture (except Kheifets), no theory has succeeded in calculating the particle distribution in the entire region of the beam-beam parameter or in agreeing with results of experiments or computer simulations. This is mainly due to improper truncation of the perturbation expansion series at the low order and/or due to use of incomplete Green's function for particle propagation which does not include full effects of particle orbit distortion by the beam-beam force. The problem, after all, hinges on obtaining the correct Green's function which gives the true transition probability of particle trajectories at any preceding moment, no matter whether particle motion is chaotic or not. Even chaotic behavior of particle motion can be described statistically and quantitatively with this correct Green's function. Then, one can calculate the particle distribution at any moment, once one knows the initial distribution. The author has proposed a new approach[4], called "the renormalized theory of beam-beam interaction," in which he borrows the renormalization technique of quantum field theory to evaluate the correct Green's function with the perturbation method. The essence of the theory may be summarized as follows. The particle distribution is decomposed into a set of modes or resonances. Their motion may not be independent of the existence of others. Actually they perturb each other mutually through the mode-coupling effect. In this interaction between resonances, there is the so-called "the self-interaction term," namely, the effect in which the change in other resonances induced by the change in a particular resonance by the mode-coupling effect acts back to the initial resonance by the mode-coupling effect to change it. There are also higher-order self-interaction processes going through many intermediate resonances before coming back to the initial resonance. If one "renormalizes" the equation for evaluation of a resonance by moving those self-interaction terms from the mode-coupling terms, the resonance as a solution of this equation will behave independent of others, since its change does not influence itself at any mean, and other resonances act only as incoherent noise sources. Now the system is diagonalized by a new set of almost orthogonal eigenfunctions. With this renormalization technique of the Green's function, the theory can always predict the correct transition probability of the particle orbit for any level of chaos to a good appromixation. This is the most significant advantage of the present theory. We can derive a new criterion for the onset of the chaotic motion from consideration of the strength of interaction between resonances. When this criterion is satisfied, the self-interaction term (renormalization correction) provides a fast diffusion of orbits, which simulates the random motion of a chaotic particle. The diffusion regions and their rates can be also calculated by the theory. The formulation of the theory and the interpretation of its physics are found in Ref. 4. In this note, we limit the explanation of the theory only to a brief summary of the final results. We rather would like to concentrate on the verification of the theory by comparison with the results of computer simulations.

Summary of the Theory

We concentrate our study on the evaluation of the averaged particle distribution $\langle P \rangle$ (averaged over the azimuthal angle ϕ in phase space) instead of that of total distribution P, and approximate the latter by the former. This is a good approximation in an experimental sense, since the microscopic fluctuating parts are usually averaged out in the measuring processes. An approximate equilibrium averaged distribution can be calculated by the following algorithm. The derivation is given in Ref. 4.

1. For all (k,v) resonances (k = even) which satisfy

$$\nu_\beta \le \frac{\nu}{k} \le \nu_\beta + \xi , \qquad (1)$$

where ν_β is the betatron tune and ξ is the beam-beam parameter, judge whether stochasticity has ensued in their vicinity, by applying the criterion [4]

$$\left(\left| \frac{U_\ell(I_{k\nu})\, \Delta\nu'(I_{k\nu})}{2\pi} \right| \right)^{1/2} > \frac{|kn - \ell\nu|}{2|k(k-\ell)|^{3/4} |\ell|^{1/2}} \qquad (2)$$

to all the possible pairs with $(k - \ell, \nu - n)$ resonances (ℓ = even). Here U_ℓ is the Fourier component of order ℓ of the beam-beam potential, $\Delta\nu'$ is the derivative of the nonlinear detuning term $\Delta\nu$ with respect to the nominal action variable I, and $I_{k\nu}$ is the resonant amplitude of (k,v) resonance.

2. Computer the additional diffusion coefficient D(I) according to the following procedure:

* This was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, High Energy Physics Division, U.S. Dept. of Energy, under Contract No. DE-AC-3-76SF0098.

$$D(I) = \sum_{k \neq 0} \sum_{\nu} \left(\frac{k}{2\pi}\right)^2 U_k^2(I)\,\mathrm{Re}\,(G_{k\nu}(I)), \tag{3}$$

where ν are integers,

$$\mathrm{Re}(G_{k\nu}) = \begin{cases} \frac{\pi}{2}\tau_{k\nu}, & \text{for } \left|\nu - k(\nu_\beta + \Delta\nu(I))\right| \le \frac{1}{\tau_{k\nu}} \\ \frac{|\gamma k/2|}{(\nu - k(\nu_\beta + \Delta\nu(I)))^2 + (\gamma k/2)^2}, & \text{for } \left|\nu - k(\nu_\beta + \Delta\nu(I))\right| > \frac{1}{\tau_{k\nu}} \end{cases} \tag{4}$$

in the stochastic regions in which the criterion (2) is satisfied, and

$$\mathrm{Re}(G_{k\nu}) = \frac{|\gamma k/2|}{(\nu - k(\nu_\beta + \Delta\nu(I)))^2 + (\gamma k/2)^2} \tag{5}$$

in the stable regions. Here

$$\tau_{k\nu} = \left(\left|\frac{U_\ell(I_{k\nu})}{2\pi} \cdot \ell\Delta\nu'(I_{k\nu})\right|\right)^{-1/2} \frac{|k-\ell|^{1/4}}{|k|^{3/4}} \tag{6}$$

is the orbit decorrelation time, and γ is defined by $\gamma = 2\gamma_y/\omega_0$ where γ_y is the linear radiation damping rate and ω_0 is the angular revolution frequency.

3. Carry out the following integration with D(I) to obtain the solution $\langle P(I)\rangle$:

$$\langle P(I)\rangle = K \cdot \exp\left[-\int_0^I \frac{1}{1 + D(I)/(\gamma\sigma^2 I)} \cdot \frac{dI}{\sigma^2}\right], \tag{7}$$

where K is a normalization factor and σ is the standard deviation of the transverse beam size.

From Eq. (7), one can see that $\langle P\rangle$ will be flattened around resonances or fast diffusion regions due to chaos where $D(I) \ge \gamma\sigma^2 I$. One should notice that these flattened regions are confined to the band-shape regions given by Eq. (4). This reflects the fact that chaotic particle can wander only in limited space as will be seen in the next section from computer simulations.

We would like to make two remarks. The Green's function (4) gives the uniform transition probability for chaotic particles inside of the chaotic region. This is apparently not true. Note that the expression (4) is only the approximated solution after some simplifications and rough assumptions. However, this may provide a good estimate of the Green's function for computing $\langle P\rangle$, since the fine structures of the true Green's function will be washed out when D is integrated in Eq. (7). In fact, the results of computer simulations confirms this statement in the next section. The second point is that the transition from regular motion to chaotic motion is treated in the final result as a rather sudden jump, whose border is separated by the criterion (2). In reality, there should be smooth transition regime. This intermediate regime is apparently mishandled in the present approximation. This defect may be removed by improving the approximation method of the Green's function.

Comparison with Computer Simulations

The computer program REBECCA (REnormalized theory of Beam-beam interaction in an Electron-positron Colliding Circular Accelerator) has been developed for computing the particle distribution $\langle P\rangle$ according to this theory. The tracking program TRACK to simulate the beam-beam interaction has been also written for comparison. The simulation technique is based on that of Ref. 5. At first, we show a nonchaotic example in which particle motion is regular in entire phase space, and then show three chaotic examples in which chaos develops amply in phase space. In all examples, γ is set to be 0.001.

Figure 1(a) shows one quarter of the phase space trajectories calculated by TRACK for $\nu_\beta = 0.22$ with $\xi = 0.04$ when the synchrotron radiation effects are turned off to see clearly that particle motion is regular. This plot is taken just in the middle of beam-beam kick so that the phase space trajectories become mirror-symmetric with respect to each coordinate. One can see the fourth-order resonance at amplitude of about one standard deviation. Figure 1(b) shows the particle distributions as a function of amplitude in polar coordinate in unit of one standard deviation σ. The solid line denotes the analytical result computed by REBECCA, while the open triangles represent the results of computer simulations. The broken line indicates the Gaussian distribution in the absence of the beam-beam force except its linear part. It is plotted as a measure to see the deviation of $\langle P\rangle$ from a Gaussian shape. A good agreement can be seen in the figure between the analytic results and those of the computer simulations.

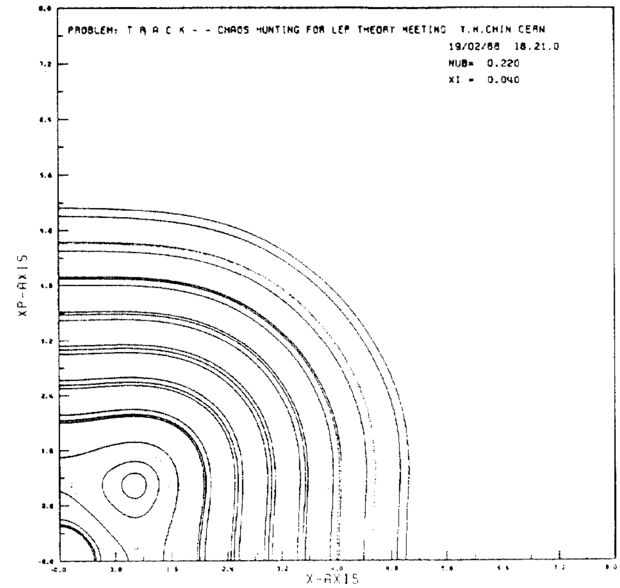

Fig. 1(a) One quarter of phase-space trajectories for $\nu_\beta = 0.22$ and $\xi = 0.04$.

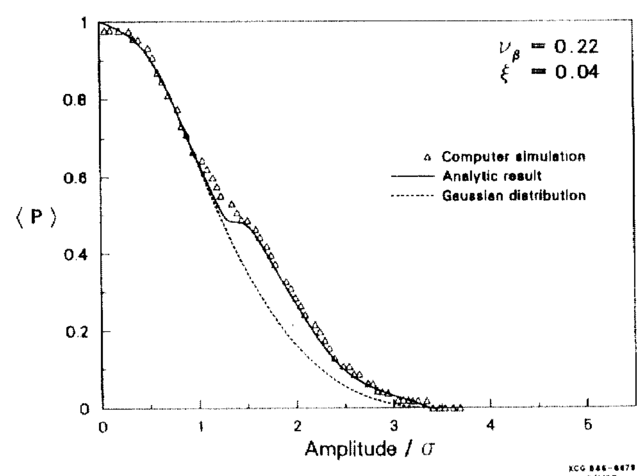

Fig. 1(b) Particle distributions as a function of amplitude. The solid line denotes the analytical result, while the open triangles represent the results of the computer simulation. The broken line indicates the Gaussian distribution in the absence of the beam-beam force except its linear part.

Comparisons for other sets of ν_β and ξ show also good agreements, unless the effect of nonchaotic resonance is so strong that particle trajectories are distorted too much to be described by the perturbation theory. But once chaos ensues, we can obtain again good agreements between the analytical results and computer simulations as seen later, since the renormalization correction to the Green's function improves the accuracy of the perturbation calculation.

Figure 2(a) shows an example of the very chaotic phase-space trajectories when the synchrotron radiation effects are turned off in order to see clearly chaotic behavior of particle. The parameters used are: ν_β = 0.15, and ξ = 0.17. The computation with REBECCA including up to the 16-th resonances says that the phase space should be chaotic at amplitude between 1.2σ and 4σ. The particle distributions are plotted in Fig. 2(b) where the same notations of lines as in Fig.1(b) are used. One can observe the reasonably good agreement between the two results except at amplitude of around 2.5σ. The hard edge of analytical distribution around there originates from the approximation of the Green's Function by the rectangle shape, although the true Green's function should have a smoother shape. This problem may be removed by improving the approximation method. Figure 3(a) shows another example of chaotic particle trajectories for ν_β = 0.08 and ξ = 0.16. The particle distributions are plotted in Fig. 3(b) in log scale, since no significant deviation from the Gaussian distribution can be seen in linear scale. The excellent agreement can be recognized.

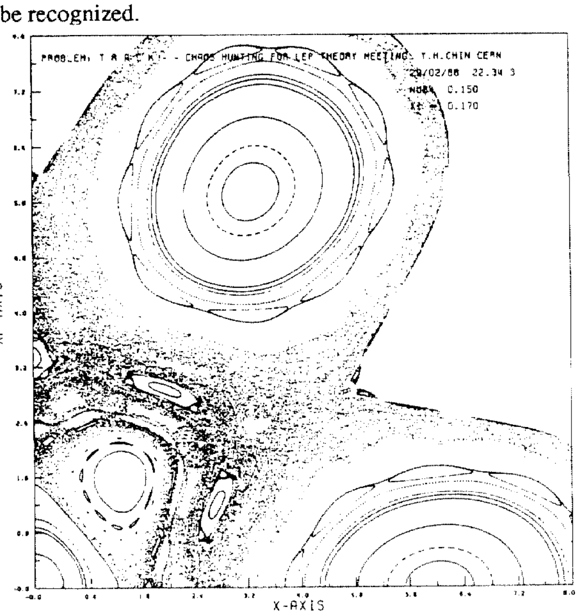

Fig. 2(a) One quarter of phase-space trajectories for ν_β = 0.15 and ξ = 0.17.

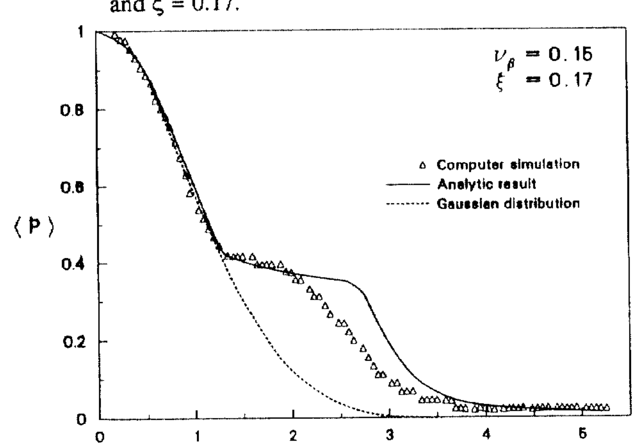

Fig. 2(b) Particle distributions for ν_β = 0.15 and ξ = 0.17.

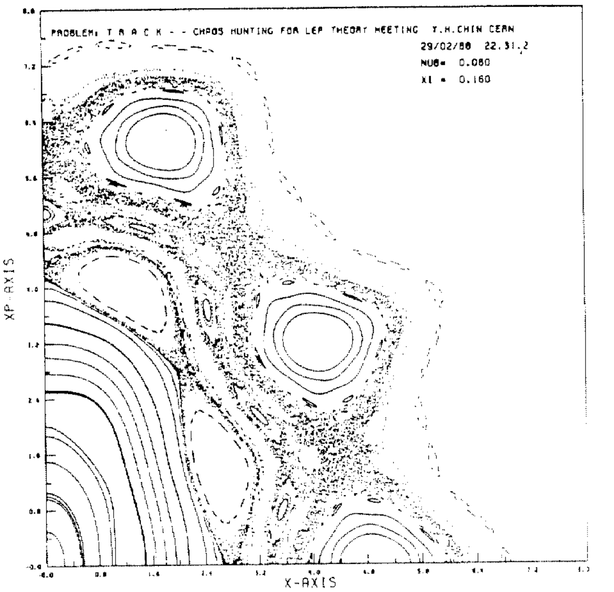

Fig. 3(a) One quarter of phase-space trajectories for ν_β = 0.08 and ξ = 0.16.

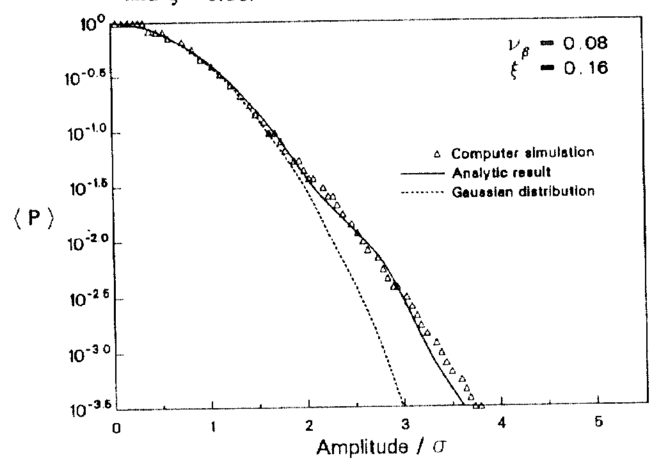

Fig. 3(b) Particle distributions for ν_β = 0.08 and ξ = 0.16.

Conclusion

We have seen that the theory shows fairly good agreements with the results of computer simulations. The present theory can predict (approximately), not only for what parameter the transition from regular motion to chaotic motion happens, but also in which region of phase space the resulting diffusion of particle orbits occurs and how large the diffusion rate is. Unlike other theories which seek only the change in r.m.s. beam size, the present theory can calculate the particle distribution as a function of amplitude. This is particularly important when one wants to estimate the beam lifetime due to boundaries such as vacuum chamber, while the computer simulation is irrelevant to such a long term process. However, the present one-dimensional strong beam-weak beam picture is still impractical for application to real machines. Further research should be made to extend the method to either the two-dimensional case or the strong-strong beams case, and eventually to both.

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