SLIM -- AN EARLY WORK REVISITED

Alex Chao

Let me start with the classic paper:

E.D. Courant and H.S. Snyder, Ann. Phys. 3, 1 (1958)

Basic idea:

- Define vector $Z = \begin{bmatrix} x \\ x' \end{bmatrix}$
- Use 2x2 transport matrices $M(1\rightarrow 2)$ to describe particle motion. $M(1\rightarrow 2)$ is calculated by multiplying matrices element by element from position 1 to position 2.

It is important to recognize that all the physics are contained in the transport matrices. Our job is to somehow extract maximum information from them. Question, is how.

Courant and Snyder's solution is to introduce a set of (now well-known) auxiliary functions:

$$\alpha(s),\ \beta(s),\ \gamma(s),\ \psi(s) \quad \text{and} \quad \eta(s),\ \eta'(s)$$

Our goal is to calculate physical quantities (e.g. closed orbit distortions, momentum compaction factor, the betatron and synchrotron tunes, the *x*-*y* coupling coefficient, the rms beam sizes, bunch length, energy spread…).

Note that this long list does not contain the auxiliary functions themselves. These functions themselves are not physical quantities.

In the Courant-Snyder tradition, we have been doing accelerator physics in three steps:

$$\boxed{\begin{array}{c}\underline{\text{Step 1}}\\ M(1\rightarrow 2)\end{array}} \rightarrow \boxed{\begin{array}{c}\underline{\text{Step 2}}\\ \alpha,\beta,\gamma,\psi,\eta,\eta',\mathcal{H}\end{array}} \rightarrow \boxed{\begin{array}{c}\underline{\text{Step 3}}\\ \text{closed orbits,}\\ \text{beam sizes, etc.}\\ \text{using } \alpha,\beta,\gamma,\psi,\eta,\eta',\mathcal{H}\end{array}}$$

Input contains
all physics

Auxiliary step.
No physics here.

Physics results

However, questions arise:

- The auxiliary functions are not physical, is Step 2 really necessary?

- Formalism applies only to 1-D, uncoupled case. To establish Step 3, we use textbook formulae, expressed in terms of the auxiliary functions. In actual applications, what replaces these formulae when the 1-D condition breaks down?

Momentum compaction factor	$\alpha_c = \frac{1}{C}\int \frac{\eta\, ds}{\rho}$
Closed orbit	$\Delta x = \frac{\theta_k \sqrt{\beta_k \beta}}{2 \sin \pi \nu_x} \cos[\pi \nu_x - \lvert\psi - \psi_k\rvert]$
Beam size	$\sigma_x^2 = \sigma_{x,\beta}^2 + \sigma_\delta^2 \eta^2$
	$\frac{\sigma_{x\beta}^2}{\beta} = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \frac{\gamma^2}{1-\mathcal{D}} \frac{\int \frac{\mathcal{H} ds}{\lvert\rho\rvert^3}}{\int \frac{ds}{\rho^2}}$
Damping partition	$J_x = 1 - \mathcal{D}, \quad J_y = 1, \quad J_s = 2 + \mathcal{D}$
	$\mathcal{D} = \frac{\int \frac{\eta ds}{\rho}(2K + \frac{1}{\rho^2})}{\int \frac{ds}{\rho^2}}$

What’s needed: to calculate the beam's physical parameters directly from the transport matrices (6x6 with general coupling) without resorting to auxiliary functions.

This is a very practical need.

- How to extend Courant-Snyder is an old topic. Many have tried.
- This talk concerns one such effort in 1979-81

  J. Appl. Phys. 50(2), 595 (1979)

  Nucl. Inst. Meth. 180, 29 (1981)
- Other important efforts:

  F. Ruggiero, E. Picasso, L. Radicati, Ann. Phys. 197, 439 (1990)

  D. Barber, K. Heinemann, H. Mais, G. Ripkin, DESY-91-146 (1991)

  K. Ohmi, K. Hirata, K. Oide, Phys. Rev. E 49, 751 (1994)

  E. Forest, Phys. Rev. E 58, 2481 (1998)

  A. Wolski, Phys. Rev. ST Accel. Beams 9, 024001 (2006)

  B. Nash, Ph.D. Thesis, Stanford University (2006)

  etc.

Courant-Snyder representation is not unique

Not being unique
=> Something there must only be an artifact.

Courant-Snyder representation (familiar):

Betatron phase

$$x(s) = \sqrt{\epsilon\beta(s)}\,\sin\psi(s), \qquad \psi(s) = \int^s \frac{ds'}{\beta(s')}$$

$$\begin{bmatrix} u \\ u' \end{bmatrix} = \begin{bmatrix} \frac{x}{\sqrt{\beta}} \\ \frac{\alpha x + \beta x'}{\sqrt{\beta}} \end{bmatrix}$$

Normalized coordinates

But this elegant formalism is not unique. Another possible representation (equally elegant) is

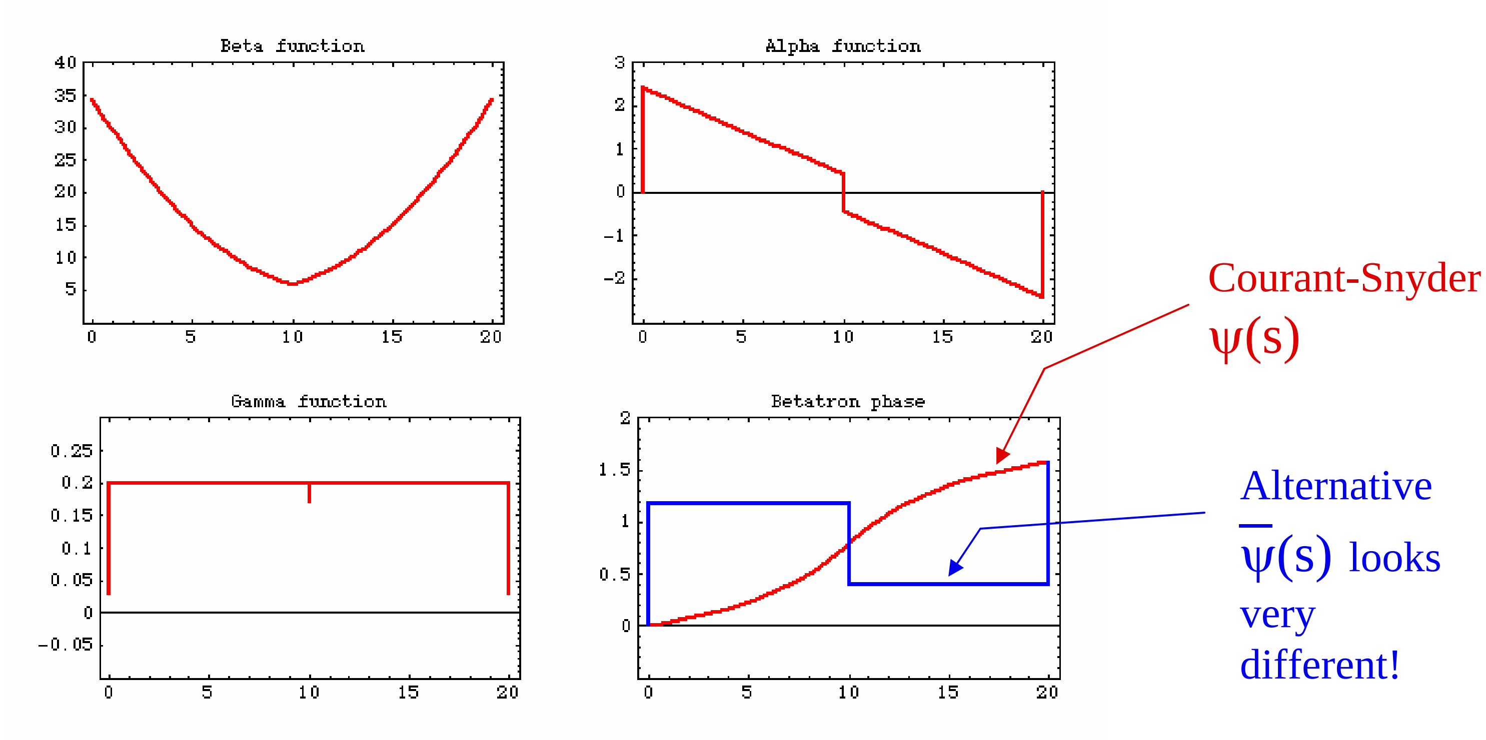
$$x'(s) = \sqrt{\epsilon\gamma(s)}\,\sin\bar{\psi}(s), \qquad \bar{\psi}(s) = \int^s \frac{K(s')ds'}{\gamma(s')}$$

An alternative betatron phase

$$\begin{bmatrix} \bar{u} \\ \bar{u}' \end{bmatrix} = \begin{bmatrix} \frac{\gamma x + \alpha x'}{\sqrt{\gamma}} \\ \frac{x'}{\sqrt{\gamma}} \end{bmatrix}$$

Alternative normalized coordinates

For example, a FODO cell:



Had we chosen the alternative representation, textbooks will all look different today. And yet, both representations give identical final results.

Replacing the auxiliary functions by eigenvectors

Replace the 3-step Courant-Snyder scheme by

$$\boxed{\begin{array}{c}\underline{\text{Step 1}}\\ M(1 \longrightarrow 2)\end{array}} \longrightarrow \boxed{\begin{array}{c}\underline{\text{Step 2}}\\ \text{eigen-vectors}\end{array}} \longrightarrow \boxed{\begin{array}{c}\underline{\text{Step 3}}\\ \text{closed orbits,}\\ \text{beam sizes, etc.}\\ \text{using eigen-vectors}\end{array}}$$

Eigenvalues and eigenvectors contain all information contained in a transport matrix (6x6, coupled) => No loss of information in Step 2.

In contrast, some information is lost if one uses Courant-Snyder representation to coupled systems.

I shall call this replacement scheme SLIM, following the name of an early computer code.

Calculating physical quantities using eigenvectors

Define $$Z = \begin{bmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \end{bmatrix}$$

Motion described by 6x6 transport matrices.

First step is to calculate the 6 eigenvalues: $e^{\pm i2\pi\nu_k}, \quad k = I, II, III$

and the 6 eigenvectors: $E_{I,II,III}$ and $E^*_{I,II,III}$

Physical quantities are then obtained from these eigenvalues and eigenvectors (the new set of auxiliary functions).

EXAMPLES

Tunes:

The 6 eigenvalues give 3 eigen-mode tunes ν_k, k=*I*,*II*,*III*.

Closed orbit:

Calculation gives a 6-D closed orbit.

Coupling effects:

Skew quadrupoles, crab cavities, solenoids, sextupoles.

Radiation damping constants:

Modifying transport matrices for rf cavities and dipole magnets to include radiation damping effects, the 6 eigenvalues become

$$e^{-\alpha_k \pm i2\pi\nu_k}, \quad k = I, II, III$$

where α_k are the eigen-mode radiation damping constants. This replaces the conventional way (valid when uncoupled) using the partition number *D*.

Beam sizes and shapes:

The 21 beam distribution moments are given by

$$\langle Z_i Z_j \rangle = 2 \sum_{k=I,II,III} \epsilon_k \, \mathrm{Re}[E_{ki} E^*_{kj}]$$

Eigenvectors instead of *H* function

$$\epsilon_k = \frac{55}{48\sqrt{3}} \frac{r_e \hbar \gamma^5}{m_e c \alpha_k} \oint ds \frac{|E_{k5}(s)|^2}{|\rho(s)|^3} \qquad k = I, II, III$$

Three eigen-emittances.

Reduce to $\varepsilon_{x,y,z}$ if 1-D uncoupled.

Spin polarization

In SLIM, we aim for a single computing framework that covers a range of situations:

- Betatron motion and synchrotron motion
- Coupled case and uncoupled case
- Near resonances and away from resonances
- Spin motion and orbital motion
- Orbital resonances and depolarization resonances

=> The original SLIM program had only 1000 lines.

Define $$Z = \begin{bmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \\ \alpha \\ \beta \end{bmatrix}$$

Two additional coordinates for spin

It is now straightforward to extend it one step more, from 3-D (x,y,z) to 4-D (x,y,z + spin). Transport matrices become 8x8. More physical quantities are computed:

<u>Spin tune:</u>

The 4-th pair of eigenvalues gives the spin precession tune ν_{IV}

It is now straightforward to extend it one step more, from 3-D to 4-D dynamics to include spin motion. Transport matrices become 8x8. More physical quantities are computed:

<u>Spin tune:</u>

The 4-th pair of eigenvalues gives the spin precession tune ν_{IV}

<u>Depolarization time and equilibrium level of polarization:</u>

These are obtained by analogy between spin motion and orbital motion:

	Diffusion		Damping	Beam property
Orbital motion	Radiation damping	$\longleftrightarrow$	Quantum excitation	Emittances
Spin motion	Radiative polarization	$\longleftrightarrow$	Spin diffusion	Polarization

Summary

After 50 years, the Courant-Snyder formalism remains as elegant and unshakable as ever.

To extend beyond 1-D applications, an eigen-analysis has been proposed to treat the general coupled cases in multi-dimensions, including the dimension of spin.