COLLISIONLESS RELAXATION IN THE TRANSPORT OF SPACE CHARGE DOMINATED BEAMS*

Yan Levin, Renato Pakter†, Tarcisio N. Teles
Insituto de Física, Universidade Federal do Rio Grande do Sul, Brazil

Abstract

In this paper, a theory is presented which allows to quantitatively predict the final stationary state achieved by a transported space-charge dominated beam during a process of collisionless relaxation. It is shown that a fully matched beam relaxes to a Fermi-Dirac distribution. However, when a mismatch is present and the beam oscillates, halo formation leads to a phase separation. The theory developed allows to quantitatively predict both the density and the velocity distributions in the final stationary state, including the halo.

INTRODUCTION

Relaxation to a final stationary state of particles interacting through long-range forces, such as the eletromagnetic interaction among particles in an intese beam, is intrinsically different than that of systems with short-range interactions, such as neutral gases and quasi-neutral plasmas, where the Coulomb interaction is screened. In the latter case, the interparticle collisions drive the system to an equilibrium state described by the Maxwell-Boltzmann distribution. This distribution is unique, in the sense that it is completely determined by the globally conserved quantities such as the total energy, momentum, angular momentum, etc. — and is otherwise independent of specific initial conditions. Relaxation of particles interacting by long-range potentials, on the other hand, is very different. For these systems, the collision duration time diverges and the state of thermodynamic equilibrium is never reached. Instead, the dynamics evolves to a stationary state in which distribution functions *appear* to stop varying with time. Unlike thermodynamic equilibrium, in such stationary state, however, detailed balance is violated and neither *equilibrium* thermodynamics nor *equilibrium* statistical mechanics can be used. Here, we investigate the relaxation of space-charge dominated beams transported through a uniform focusing field.

MODEL

We consider the transport of intense, continuous, charged-particles beams through a uniform focusing magnetic field. The beam is assumed to propagate with a constant axial velocity $v_z\hat{\mathbf{e}}_z$, so that the axial coordinate $s = z = v_z t$ plays the role of time and to be symmetric with respect to the $z - axis$. The external focusing field is given by $\mathbf{B} = B_o\hat{\mathbf{e}}_z$ and is used to compensate the repulsive Coulomb force between the beam particles. It is convenient to work in the Larmor frame of reference [1], which rotates with respect to the laboratory frame with angular velocity $\Omega_L = qB_o/2\gamma_b mc$, where c is the speed of light in *vacuo*, and q, m, and $\gamma_b = [1 - (v_z/c)^2]^{-1/2}$ are the charge, mass, and relativistic factor of the beam particles, respectively. In this frame, the beam distribution function $f_b(s, \mathbf{r}, \mathbf{v})$ evolves according to the Vlasov-Poisson system [1]

$$\frac{\partial f_b}{\partial s} + \mathbf{v}\cdot\nabla f_b + (-\kappa_z\mathbf{r} - \nabla\psi)\cdot\nabla_{\mathbf{v}} f_b = 0, \quad (1)$$

$$\nabla^2\psi = -(2\pi K/N_b)\, n_b(\mathbf{r}, s), \quad (2)$$

where N_b is the number of particles per unit axial length, $\mathbf{r}$ is the position vector in the transverse plane, and $\mathbf{v} \equiv d\mathbf{r}/ds$ is the transverse velocity, $n_b(\mathbf{r}, s) = N_b \int f_b\, \mathrm{d}^2\mathbf{v}$ is the transverse beam density profile, $\kappa_z = q^2 B_o^2/4\gamma_b^2 v_z^2 m^2 c^2$ is the focusing field parameter, and $K = 2q^2 N_b/\gamma_b^3 v_z^2 mc^2$ is the beam perveance, which is a measure of the beam intensity. In Eqs. (1) and (2), ψ is a scalar potential that incorporates both self-electric and self-magnetic fields, $\mathbf{E}^s$ and $\mathbf{B}^s$. We shall take zero of the scalar potential to be at r_w, the position of the conducting channel wall. The distribution function is normalized, so that $\int f_b\, \mathrm{d}^2\mathbf{r}\, \mathrm{d}^2\mathbf{v} = 1$. We assume that the initial beam distribution corresponds to an uncorrelated uniform distribution in both space and velocity,

$$f_{b0}(\mathbf{r}, \mathbf{v}) = \eta_1 \Theta(r_m - r)\Theta(v_m - v) \quad (3)$$

with $\eta_1 = 1/\pi^2\varepsilon_0$, where $\varepsilon_0 = r_m^2 v_m^2$ is the initial unnormalized emittance of the beam and $Theta$ is the Heaviside step funcion. The distribution function Eq. (3) is not an stationary solution of the Vlasov-Poisson system, and for $s > 0$ the system will start to evolve. Our aim is to determine the final stationary state attained by the beam.

In order to verify the theoretical findings to be discussed, we also perform self-consistent particle simulations. In the simulations, $N = 5000$ macroparticles are launched according to the prescribed initial distribution and evolve by interacting with the other beam macroparticles and with the focusing magnetic the field. Taking advantage of the axisymmetry of the beam, we can easily compute the beam force on a given particle using Gauss law. The dynamics of the ith macroparticle is dictated by

$$\frac{d^2 r_i}{ds^2} + \kappa_z r_i - \frac{N_i}{N}\frac{K}{r_i} - \frac{P_{\theta i}^2}{r_i^3} = 0, \quad (4)$$

* Work supported by CNPq and FAPERGS, Brazil, and by the US AFOSR, grant n. FA9550-06-1-0345.
† pakter@if.ufrgs.br.

where $N_i = \sum_{j\neq i} \Theta(r_i - r_j)$ is the number of macorparticles with radius smaller that r_i, and $P_{\theta i}$ =const. is the conserved particle angular momentum.

THEORY OF VIOLENT RELAXATION

Vlasov equation (1) shows that the distribution function evolves in time as an incompressible fluid. While the fine-grained distribution function $f_b(s, \mathbf{r}, \mathbf{v})$ never reaches a stationary state – the evolution continues on smaller and smaller length scales *ad infinitum* – Lynden-Bell [2] argued that the *coarse-grained* distribution function $\bar{f}_b(s, \mathbf{r}, \mathbf{v},)$, *averaged* on microscopic length scales, will rapidly relax to a meta-equilibrium with $\bar{f}_b(\mathbf{r}, \mathbf{v})$. For gravitational systems Lynden-Bell called this process "a violent relaxation". To obtain the stationary distribution $\bar{f}_b(\mathbf{r}, \mathbf{v})$, we divide the phase space into macrocells of volume $\mathrm{d}^d\mathbf{r}\,\mathrm{d}^d\mathbf{v}$, which are in turn subdivided into ν microcells, each of volume h^d. As a consequence of incompressibility, each microcell can contain at most one discretized level η_j. The number density of the level j inside a *macrocell* at $(\mathbf{r}, \mathbf{v})$ — number of microcells occupied by the level j divided by ν — will be denoted by $\rho_j(\mathbf{r}, \mathbf{v})$. Note that by construction, the total number density of *all* levels in a macrocell is restricted to be $\sum_j \rho_j(\mathbf{r}, \mathbf{v}) \leq 1$.

Using a standard combinatorial procedure [2] it is then possible to associate a coarse-grained entropy with the distribution of $\{\rho_j\}$. Lynden-Bell argued that collisionless relaxation should lead to the density distribution of levels which is the most likely, i.e. the one that maximizes the *coarse-grained* entropy, consistent with the conservation of all the dynamical invariants — energy, momentum, angular momentum and the hypervolumes $\gamma(\eta_j)$. In terms of the number densities $\{\rho_j\}$ which maximize the coarse-grained entropy, the stationary distribution function becomes a Fermi-Dirac distribution,

$$\bar{f}_b(\mathbf{r}, \mathbf{v}) = \eta_1 \rho(\mathbf{r}, \mathbf{v}) = \frac{\eta_1}{e^{\beta[\epsilon(\mathbf{r},\mathbf{v})-\mu]} + 1}\ , \tag{5}$$

where ϵ is the mean energy of particles with velocity $\mathbf{v}$ at position $\mathbf{r}$, and β and μ are the two Lagrange multipliers required by the conservations of energy and number of particles,

$$\int \mathrm{d}^d\mathbf{r}\,\mathrm{d}^d\mathbf{v}\ \epsilon(\mathbf{r}, \mathbf{v})\bar{f}_b(\mathbf{r}, \mathbf{v}) = \epsilon_0 \tag{6}$$

$$\int \mathrm{d}^d\mathbf{r}\,\mathrm{d}^d\mathbf{v}\bar{f}_b(\mathbf{r}, \mathbf{v}) = 1\ .$$

In the above formula ϵ_0 is the energy per particle specified by the original distribution f_{b0}. For an azimuthally symmetric system, the mean particle energy ϵ is a function of only the modulus r and v. By analogy with the usual Fermi-Dirac statistics, we define $\beta = 1/k_B T$, were T is the effective temperature of the stationary state (not to be confused with the usual definition of temperature in terms of the average kinetic energy which is valid only for classical systems in thermodynamic equilibrium) and μ is the effective beam chemical potential. The maximum entropy state, however, can only be achieved if there is a sufficient ergodicity (mixing) in the phase space.

RMS MATCHED BEAMS

It is possible to adjust the values of r_m and v_m in such a way that during the evolution, the beam envelope (rms beam size) oscillates as little as possible. This corresponds to the so called matched beam condition — the beam relaxes to equilibrium, but without undergoing significant macroscopic oscillations. The matching condition can be determined using beam envelope equation [1]. For the distribution function (3), it is possible to show that a beam is matched if

$$v_m^2 \approx \kappa_z r_m^2 - K. \tag{7}$$

When this condition is met, we expect the mixing to be efficient and Lynden-Bell theory to apply. The coarse-grained beam distribution should then relax to Eq. (5), with $\epsilon(r, v) = v^2/2 + U(r) + \psi(r)$, where the mean electrostatic potential $\psi(r)$ is determined self-consistently by an iterative solution of Eq.(2), subject to constraints of Eqs. (6) with energy per particle given by

$$\epsilon_0 = \frac{v_m^2}{4} + \frac{\kappa_z r_m^2}{4} + \frac{1}{8} - \frac{K}{2}\ln\left(\frac{r_m}{r_w}\right)\ . \tag{8}$$

To compare the theory with the simulations, we calculate the number particles inside shells located between r and $r + \mathrm{d}r$, $N(r)\mathrm{d}r = 2\pi N_b r \mathrm{d}r \int \mathrm{d}^2\mathbf{v}\bar{f}(\mathbf{r}, \mathbf{v})$; and the number of particles with velocities between v and $v + \mathrm{d}v$, $N(v)\mathrm{d}v = 2\pi N_b v \mathrm{d}v \int \mathrm{d}^2\mathbf{r}\bar{f}(\mathbf{r}, \mathbf{v})$. In Fig. 1 the solid lines show the values of $N(r)/N_b$ and $N(v)/N_b$ obtained using the theory described above, while points are the result of a self-consistent N-particle dynamics simulation. In all the figures distances are measured in units of $\sqrt{K/\kappa_z}$ and velocities in units of $\sqrt{K}$. Good agreement between the theory and the simulation is found for both position and velocity distributions *without* any fitting parameters. We have checked that agreement persists for other values of r_m and v_m, as long as the matching condition (7) is satisfied.

MISMATCHED BEAMS

Macroscopic beam oscillations lead to a number of important consequences which are not taken into account in the theory of violent relaxation. In particular, the oscillations excite parametric resonances [3] transferring large amounts of energy to some particles at the expense of the rest [4]. This mechanism leads to inefficient phase space mixing and non-ergodicity. As the relaxation proceeds, the oscillating beam core becomes progressively colder, while a halo of highly energetic particles is created – a sort of evaporative cooling process. However, because of the incompressibility restriction imposed by the Vlasov dynamics, the core can not freeze – collapse to the minimum of the potential energy. Instead, the distribution function of

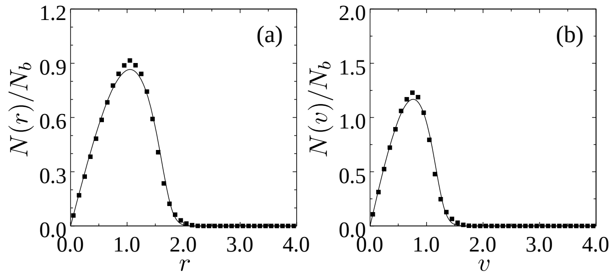

Figure 1: Position and velocity distributions for a matched beam with $r_m = 1.48\sqrt{K/\kappa_z}$ and $v_m = 1.1\sqrt{K}$. Solid line is the theoretical prediction obtained using distribution function of Eq. (5), while points are the result of the dynamics simulation with $N = 5000$ particles.

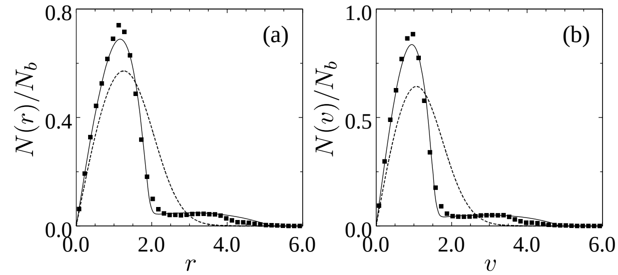

Figure 2: (a) Position and (b) velocity distributions. Points are the result of dynamics simulations. Solid curves are the theoretical predictions obtained using the distribution function of Eq. (9). Dashed curves are the predictions of the violent relaxation theory based on Eq. (5). The initial distribution is uniform with $r_m = 1.0\sqrt{K/\kappa_z}$ and $v_m = 2.4\sqrt{K}$.

the core particles progressively approaches that of a fully degenerate Fermi gas.

The extent of the halo is determined by the location of the parametric resonance, and its range r_R can be calculated either using the canonical perturbation theory [3] or test particle analysis [4, 5]. The first resonant particles move in an almost simple harmonic motion with energy $\epsilon_R = K\ln(r_w/r_R) + \kappa_z r_R^2/2$. As more and more particles are ejected from the beam core their motion, however, becomes chaotic and a halo distribution becomes smeared out. We find that the distribution function of a completely relaxed halo is very well approximated by the Heaviside step function $\Theta(\epsilon_R - \epsilon)$.

For an out of (thermodynamic) equilibrium system, there are no clear parameters which will control the core-halo coexistence. We can not, therefore, *a priory* say when the halo formation will stop and a stationary state be established. Empirically, however, we have observed that this happens when the core temperature becomes sufficiently low. In all cases studied, we find that the core-halo equilibrium is achieved when the ratio between the core temperature and the corresponding Fermi temperature is $T/T_F \approx 1/40$ – i.e. when $\beta\mu \approx 40$. The stationary distribution function for the core-halo system, then, takes a very simple form [6]

$$\bar{f}_b(\mathbf{r},\mathbf{v}) = \frac{\eta_1}{e^{\beta\epsilon(\mathbf{r},\mathbf{v})-40}+1} + \chi\Theta(\epsilon_R - \epsilon)\ . \tag{9}$$

Since all the dependence on $\mathbf{r}$ and $\mathbf{v}$ enters only implicitly through ϵ, f_b automatically satisfies the Vlasov-Poisson system. The value of $\eta_1 = 1/\pi^2\varepsilon_0$, is determined by the initial distribution f_{b0}, while the value of ϵ_R is calculated from the location of the parametric resonance. This leaves to determine self-consistently, using Eqs. (6) and (2), the mean electrostatic potential $\psi(r)$, the inverse temperature β, and the amplitude χ which will determine the fraction of particles inside the halo. These can, once again, be obtained iteratively. In Fig. 2 we plot $N(r)/N_b$ and $N(v)/N_b$, obtained using the theory presented above for a mismatched beam case, and compare these distributions with the ones obtained using the dynamics simulations. Good agreement is found. For comparison, we also present in Fig. 2 the distribution functions obtained using the violent relaxation theory, Eq. (5). It is clear that this theory is unable to describe relaxation of initially mismatched beams.

CONCLUSIONS

We have studied the relaxation process for space-charge domintaded beams transported along a focusing magnetic field. Unlike normal gases with short range forces, space-charge dominated beams do not evolve to the state of thermodynamic equilibrium. Instead collisionless relaxation culminates in a stationary state in which the detailed balance is violated. Using a combination of non-equilibrium statistical mechanics and the theory of parametric resonances it is, nevertheless, possible to *a priory* predict the distribution functions for the final stationary state. Unlike the normal thermodynamic equilibrium, this state, however, explicitly depends on the initial distribution of particle velocities and positions.

REFERENCES

[1] R.C. Davidson and H. Qin, *Physics of Intense Charged Particle Beams in High Energy Accelerators* (World Scientific, Singapore, 2001).

[2] D. Lynden-Bell, Mon. Not. R. Astron. Soc. **136**, 101 (1967).

[3] R.L. Gluckstern, Phys. Rev. Lett. **73**, 1247 (1994).

[4] R. P. Nunes, R. Pakter, and F. B. Rizzato, Phys. Plasmas, **14**, 023104 (2007).

[5] T. P. Wangler, K. R. Crandall, R. Ryne, and T. S. Wang, Phys. Rev. ST Accel. Beams **1**, 084201 (1998).

[6] Y. Levin, R. Pakter, and T.N. Teles, Phys. Rev. Lett., **100**, 040604 (2008).