IMPACT OF BETATRON MOTION ON PATH LENGTHENING AND MOMENTUM APERTURE IN A STORAGE RING

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INTRODUCTION

In the beam dynamics of an electron storage ring the amplitude of the betatron motion can become so large as not to be ignored, e.g. Touschek scattered particle or injection beam. Specifically the path lengthening by the finite amplitude betatron motion can give a serious influence on beam dynamics. By means of the synchrotron motion, the variation of the path lengthening is converted into the energy deviation, so that the momentum deviation is enhanced by the impact of the betatron motion, which inevitably effects on the momentum acceptance.

It is well known that the finite amplitude of the betatron motion gives rise to the path lengthening of the central trajectory [1, 2, 3, 4]. The resulting formula of the path lengthening is simply represented by the product of the invariant amplitude and the chromaticity. In this paper the path lengthening by a finite amplitude betatron motion is derived by means of the canonical perturbation method [5, 6]. Through this derivation we elucidate that the path lengthening is produced by the central orbit shift by the non-linear betatron motion owing to the sextupole magnet potential.

Using the formula, we discuss the impact of the finite amplitude betatron motion on momentum aperture in a Touschek effect. It has been observed that the momentum acceptance becomes smaller as the chromaticity grows larger. Furthermore, the decrease of the momentum acceptance by the horizontal chromaticity growth is lager than that by the vertical. This difference can be explained by the impact of the finite amplitude betatron motion on the momentum aperture, since the scattered particle oscillates initially in a horizontal plane, and then the path lengthening is sensitive to the horizontal chromaticity.

PATH LENGTHENING BY FINITE AMPLITUDE BETATRON MOTION

Here we consider the betatron motion with a sextupole perturbation in two degree of freedom. The Hamiltonian H, describing the motion of a particle in a circular accelerator, is given by

$$\begin{aligned} &H\left(x,p_x,y,p_y,s\right) = \\ &\frac{1}{2}\left(p_x^2+p_y^2\right)+\frac{1}{2}\left[K_x^2\left(s\right)+g_0\left(s\right)\right]x^2-\frac{1}{2}g_0\left(s\right)y^2 \\ &+\frac{1}{3!}g_1\left(s\right)\left(x^3-3xy^2\right)+\frac{1}{2}K_x\left(s\right)x\left(p_x^2+p_y^2\right), \end{aligned} \quad (1)$$

where s is the path length along the reference orbit. In addition, K_x is the horizontal curvature, and $g_{0,1}$'s are the field strengths of quadrupole and sextupole magnets, respectively. Note that the momenta $p_{x,\,y}$ and $g_{0,1}$'s are normalized by the nominal momentum p_0.

The Hamiltonian consisting of the quadratic terms describes the linear betatron motion, which is represented in terms of the action-angle variables (J_z, ϕ_z) for $z = x, y$ as

$$z\left(s\right) = \sqrt{2J_z\beta_z\left(s\right)}\cos\phi_z, \quad (2)$$

where β_z is the betatron function. The cubic terms are treated as a perturbation potential, which generate non-linear oscillation in betatron motion. Although the central trajectory of the linear betatron motion is properly $\langle x\left(s\right)\rangle = 0$, where the brackets $\langle\cdot\rangle$ represents the average over the angle variable, the perturbation potential gives the shift of the central trajectory owing to the non-linear oscillation. The integral representation is derived by using the canonical perturbation method [5, 6],

$$\begin{aligned} \langle x\left(s\right)\rangle = &-\frac{J_x\beta_x^{1/2}\left(s\right)}{4\sin\pi\nu_x}\int_s^{s+C}d\bar{s}\beta_x^{1/2}\left(\bar{s}\right) \\ &\times\left[g_1\left(\bar{s}\right)\beta_x\left(\bar{s}\right)+K_x\left(\bar{s}\right)\gamma_x\left(\bar{s}\right)\right]\cos\Psi_x\left(\bar{s},s\right) \\ &-\frac{J_y\beta_x^{1/2}\left(s\right)}{4\sin\pi\nu_x}\int_s^{s+C}d\bar{s}\beta_x^{1/2}\left(\bar{s}\right) \\ &\times\left[-g_1\left(\bar{s}\right)\beta_y\left(\bar{s}\right)+K_x\left(\bar{s}\right)\gamma_y\left(\bar{s}\right)\right]\cos\Psi_x\left(\bar{s},s\right) \\ &-\frac{J_x\beta_x^{1/2}\left(s\right)}{2\sin\pi\nu_x}\int_s^{s+C}d\bar{s}K_x\left(\bar{s}\right)\beta_x^{-1/2}\left(\bar{s}\right)\alpha_x\left(\bar{s}\right) \\ &\times\left[\alpha_x\left(\bar{s}\right)\cos\Psi_x\left(\bar{s},s\right)+\sin\Psi_x\left(\bar{s},s\right)\right]. \end{aligned} \quad (3)$$

Here $\alpha_{x,y}$ and $\gamma_{x,y}$ are the Twiss parameters, and ν_x the horizontal betatron tune, and $\Psi_x\left(\bar{s},s\right) = \psi_x(\bar{s}) - \psi_x(s) - \pi\nu_x$ with $\psi_x(s) = \int_{s_0}^{s} d\bar{s}/\beta_x\left(\bar{s}\right)$. Equation (3) explicitly implies that the shift of the central orbit is produced by the sextupole potential as well as the higher order terms of the curvature K_x.

Up to the second order the variation of the average path length due to the betatron motion is approximated as

$$\Delta C = \int_0^C ds\left[K_x\langle x\rangle+\frac{1}{2}\left\langle x'^2\right\rangle+\frac{1}{2}\left\langle y'^2\right\rangle\right]. \quad (4)$$

Inserting the representation of the barycenter of the betatron motion (3) into the linear term in the above equation, we obtain the variation of the average path length due to the shift of the central orbit

$$\begin{aligned} &\int_0^C dsK_x\left(s\right)\langle x\left(s\right)\rangle = \\ &-\frac{1}{2}J_x\int_0^C ds\left(\beta_xg_1\eta_x-2\alpha_xK_x\eta_x'+\gamma_xK_x\eta_x\right) \\ &-\frac{1}{2}J_y\int_0^C ds\left(-\beta_yg_1\eta_x+\gamma_yK_x\eta_x\right). \end{aligned} \quad (5)$$

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In the above derivation of the path lengthening by the finite amplitude betatron oscillation, we use the integral representation of the dispersion function:

$$\eta_x(\bar{s}) = \frac{\sqrt{\beta_x(\bar{s})}}{2\sin\pi\nu_x}\int_{\bar{s}}^{\bar{s}+C} ds K_x(s)\sqrt{\beta_x(s)}\cos\Psi_x(s,\bar{s}). \tag{6}$$

Thus we find that the first order variation of the path length due to the betatron motion comes from the the shift of the barycenter of the nonlinear oscillation. On the other hand, the quadratic terms of the path lengthening represent the extension of the trajectory by the excursion from the central orbit, whose lowest order contribution is given by

$$\frac{1}{2}\int_0^C ds\left[\langle x'^2\rangle + \langle y'^2\rangle\right] = \frac{1}{2}\int_0^C ds\,(J_x\gamma_x + J_y\gamma_y). \tag{7}$$

Combining the above results, we obtain the path lengthening due to the finite amplitude betatron motion:

$$\Delta C = -2\pi\,(\xi_x J_x + \xi_y J_y), \tag{8}$$

where $\xi_{x,y}$'s are the linear chromaticity [7, 8].

Through the synchrotron oscillation the path lengthening is converted into the momentum deviation $\delta = \Delta p/p$: i.e. in the first order

$$\delta = \alpha_1^{-1}\frac{\Delta C}{C}, \tag{9}$$

where α_1 is the linear momentum compaction factor. Now we estimate the momentum deviation in the Touschek scattered particle at the SPring-8 storage ring. The momentum acceptance of the SPring-8 storage ring is about 3 %. If the intra-beam collision with energy change 3 % occurs at the maximum dispersion 0.3 m, where the horizontal betatron function takes 25 m, the invariant amplitude J_x is 1.6×10^{-6} m. Then the path lengthening results in the momentum deviation $4.2\times10^{-5}\xi_x$ with the momentum compaction 1.67×10^{-4} and the circumference 1436 m. Even in the case of $\xi_x \simeq 10$, the consequent momentum deviation caused by the Touschek scattering amounts to 4.2×10^{-4}, which is negligible small compared to the momentum acceptance 3 %. But, for a particle with momentum deviation 3 %, we cannot ignore the non-linearity of the chromaticity. Then the effective chromaticity $\xi_x[\equiv(\partial\nu_x)/(\partial\delta)]$ in Eq. (8) should be replaced with

$$\xi_z \approx \xi_z^{(1)} + 2\delta\xi_z^{(2)}, \tag{10}$$

where $\xi_z^{(1)}$ and $\xi_z^{(2)}$ are the linear and the second order chromaticities, respectively. Since the second order chromaticity $\xi_x^{(2)}$ is about 300 at the present case of $\xi_x^{(1)} \simeq 10$, the effective chromaticity changes to 30, and hence the momentum deviation by the finite amplitude betatron motion is 1×10^{-3}, which brings perceptible influence in the beam dynamics compared to the momentum acceptance 3 %. On the other hand, the impact of the vertical chromaticity is less influential than the horizontal, since the oscillation of a Touschek scattered particle or an injecting beam is in the horizontal plane.

MOMENTUM ACCEPTANCE AT THE SPRING-8 STORAGE RING

Momentum aperture is most directly measured through the Touschek lifetime, which is originally defined by the effect that the particle with energy exchange through the intra-beam collision spills out the rf bucket. In the SPring-8 storage ring, the Touschek lifetime can be measured under the high bunch current condition of 1 mA/bunch. To measure the momentum aperture, we measure the Touschek lifetime with changing rf voltage as shown in Fig. 1. In Fig. 1, the dotted line denotes the expected Touschek lifetime if it was limited only by the longitudinal acceptance. In practice the momentum aperture is also restricted by the transverse dynamics, since the Touschek scattered particle at a dispersive section starts to oscillate in a horizontal direction. Then the Touschek lifetime reaches the ceiling by the transverse aperture limit as the solid line in Fig. 1.

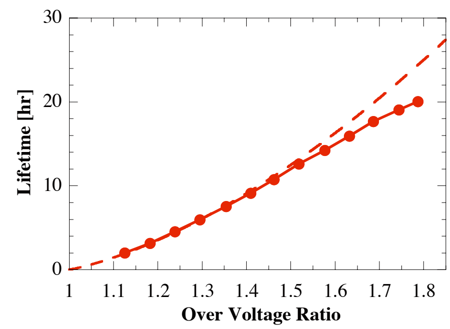

Figure 1: Touschek lifetime vs rf voltage.

To understand the particle dynamics that restricts the momentum acceptance, we investigate the dependence of the Touschek lifetime on the in-vacuum undulator gap, which is used as a vertical beam scraper. We investigate the Touschek beam lifetime for three in-vacuum undulators, i.e. long one of 25 m, and two standard of 4.5 m, shown in Fig. 2. The beam lifetime of the long undulator is

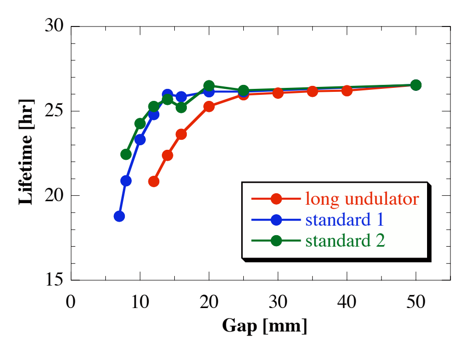

Figure 2: Touschek lifetime vs undulator gap.

different from the others. But we found that, after normalizing the gaps by the square root of the betatron function at the ends of the undulators, the beam lifetimes coincide each other. In addition, the normalized gap where the beam lifetime starts to be reduced is equal to the minimum normalized height of vacuum chamber. These facts imply that the momentum acceptance is determined by the transverse dynamics, i.e. the resonance coupling. The operation point (40.15, 18.35) of the SPring-8 storage ring is sufficiently far from the linear resonance, so that it is expected that the non-linear resonance brings the coupling.

By the way, the optics of the SPring-8 storage ring is changed from the achromat one to the distributed dispersion to reduce the emittance. Although the optics change brings the bunch volume reduction by half, the Touschek beam lifetime does not become so short as shown in Fig. 3. This is because the momentum acceptance of the distributed dispersion optics is larger than that of the achromat. Due to the small peak of the dispersion function of the former optics the amplitude of the Touschek scattered electron becomes small compared to that of the latter. Hence the vertical excursion of the scattered particle by the non-linear resonance coupling is suppressed, or the momentum acceptance conversely becomes large. This fact again implies that the momentum acceptance of the Spring-8 storage ring is restricted by the transverse dynamics of the off-momentum particle.

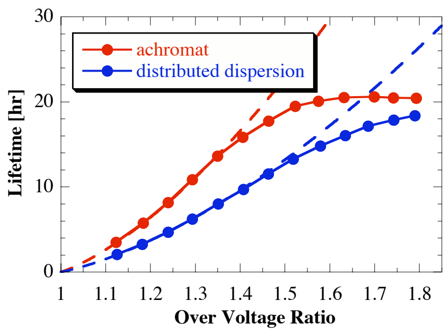

Figure 3: Touschek lifetime on ring optics.

Since the chromaticity is one of the important parameter in dynamics of a circular accelerator, we investigate its influence on the momentum acceptance. Figure 4 shows the results of the momentum acceptance measurement with changing the horizontal and the vertical chromaticities independently. The larger the chromaticities in both the directions become, the shorter the Touschek lifetime does. This is because the chromaticity excites the non-linear resonance coupling, or correctly the sidebands, so that the vertical beam spread grows larger as the chromaticity does. The growth of the vertical beam spread results in the shortage of the Touschek lifetime, i.e. the reduction of the momentum acceptance. See Fig. 5. From these results we find that the vertical chromaticity has the less impact on the momentum acceptance than the horizontal.

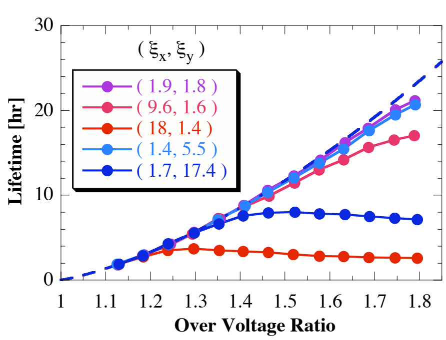

Figure 4: Normalized Touschek lifetime on chromaticity.

The cause that the impacts of the chromaticities on the Touschek lifetime are different is attributed to the path lengthening by the finite amplitude of the betatron oscillation. Since the scattered particle by the Touschek effect begins to oscillation in the horizontal plane, the average path lengthening of the scattered particle in the larger horizontal chromaticity optics may become large, and then it can enhance the momentum deviation of the scattered particle, or reduce the momentum acceptance. On the other hand, the vertical chromaticity give less impact on the momentum acceptance due to the small vertical amplitude of the scattered particle.

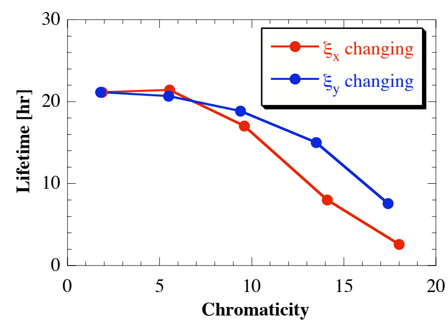

Figure 5: Touschek lifetime on chromaticity.

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