SUM OF EMITTANCES IN THE PRESENCE OF A LINEAR COUPLING

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Abstract
In the presence of a linear coupling due to skew quadrupoles the sum of the single particle horizontal and vertical emittances has carefully been obtained. It is shown that only under certain condition the sum is an invariant quantity.

INTRODUCTION

Linear coupling of horizontal and vertical oscillations is of prime importance for the operation and performance of a synchrotron [1-3]. In the presence of a linear coupling the invariance of emittances in horizontal and vertical planes no longer holds and it is a common consensus that the sum of emittances is an invariant quantity. Here we have tried to show the extent under which the sum is an invariant quantity.
We first bring the definition of the transverse single particle emittances using the Floquet transformation in alternating gradient as well as constant focusing rings, then in the presence of the linear coupling, due to skew quadrupoles we introduce the coupled differential equations governing the particles motion to find an equation for the sum of the emittances.

BASIC EQUATIONS

We first recall the usual definition of the single particle emittance

$$\varepsilon_x = \gamma_x x^2 + 2\alpha_x x x' + \beta_x x'^2, \quad (1)$$

in which β_x is the amplitude function, α_x the correlation function $\alpha_x = -\frac{\beta'_x}{2}$ and $1+\alpha_x^2 = \beta_x\gamma_x$ with a similar relation for the ε_y. For the general case of periodic focusing, the general trajectory equation of motion is

$$\frac{d^2x}{ds^2} + K(s)x = 0 \quad (2)$$

The solution to the trajectory equation would be

$$x = \sqrt{\varepsilon_x\beta_x}\cos(\Phi(s)+\psi). \quad (3)$$

From Eqs. (2) and (3) we can also show that

$$\alpha'_x = \beta_x K - \gamma_x. \quad (4)$$

We may define the following Floquet transform from (x,s) to (η,ϕ)

$$\eta = \frac{x}{\sqrt{\beta_x}} \text{ and } \phi = \frac{\Phi(s)}{Q_x} = \frac{1}{Q_x}\int\frac{ds}{\beta_x}. \quad (5)$$

We also keep in mind that

$$x' = -\frac{\sqrt{\varepsilon_x}}{\sqrt{\beta_x}}\left(\alpha\cos(\Phi(s)+\psi)+\cos(\Phi(s)+\psi)\right). \quad (6)$$

Now, the Courant-Snyder invariant Eq. (1), which corresponds to an ellipse in (x,x') phase space, with a shape and orientation which is a function of s can be cast into a new form as

$$\eta(\phi) = \frac{x}{\sqrt{\beta_x}} = \sqrt{\varepsilon_x}\cos(Q_x\phi+\psi). \quad (7)$$

$$\dot\eta(\phi) = \frac{d\eta}{d\phi} = -Q_x\sqrt{\varepsilon_x}\sin(Q_x\phi+\psi). \quad (8)$$

Therefore, we can readily write

$$\varepsilon_x = \eta^2 + \left(\frac{\dot\eta}{Q_x}\right)^2. \quad (9)$$

and similarly for ε_y. Also from a different point of view we may write from Eq. (7)

$$x' = \frac{d}{ds}\left(\eta\sqrt{\beta_x}\right) = \sqrt{\beta_x}\frac{d\eta}{ds} - \eta\frac{\alpha_x}{\sqrt{\beta_x}} = \sqrt{\beta_x}\frac{d\eta}{d\phi}\frac{d\phi}{ds} - \eta\frac{\alpha_x}{\sqrt{\beta_x}} = \frac{1}{Q_x\sqrt{\beta_x}}(\dot\eta - \eta\alpha_x Q_x), \quad (10)$$

and for the second derivative we can write

$$x'' = \frac{\ddot\eta - Q_x^2\eta(\alpha_x^2+\beta\alpha'_x)}{Q_x^2\beta_x^{3/2}}. \quad (11)$$

Substituting the above relation in Eq. (2) we are left with

$$x'' + Kx = \frac{\ddot\eta - Q_x^2\eta\left(\alpha_x^2+\beta_x\alpha'_x - K\beta_x^2\right)}{Q_x^2\beta_x^{3/2}} = 0. \quad (12)$$

Now, from Eq. (4) we note that the expression in the parenthesis is equal to −1 and therefore, we can write

$$\frac{d^2\eta}{d\phi^2}+Q_x^2\eta=0. \tag{13}$$

If we now multiply both sides by $\frac{d\eta}{d\phi}$, we will have

$$\frac{d\eta}{d\phi}\frac{d^2\eta}{d\phi^2}+Q_x^2\eta\frac{d\eta}{d\phi}=0, \tag{14}$$

or

$$\frac{d}{d\phi}\left(\frac{1}{2}\left(\frac{d\eta}{d\phi}\right)^2\right)+\frac{Q_x^2}{2}\frac{d}{d\phi}\eta^2=0, \tag{15}$$

therefore

$$\frac{d}{d\phi}\left(\frac{1}{2}\left(\frac{d\eta}{d\phi}\right)^2+\frac{Q_x^2}{2}\eta^2\right)=0. \tag{16}$$

$$\frac{1}{2}\left(\frac{d\eta}{d\phi}\right)^2+\frac{Q_x^2}{2}\eta^2=\text{const.} \tag{17}$$

$$\dot{\eta}^2+Q_x^2\eta^2=\text{const.} \tag{18}$$

$$\eta^2+\frac{\dot{\eta}^2}{Q_x^2}=\text{const.} \tag{19}$$

For the case of smooth approximation and constant focusing, $\beta_x=\text{const.}$ therefore,

$\alpha=0$ and $\beta_x=\frac{1}{\gamma_x}$. With the following change of variable $\eta=x\beta_x^{-1/2},\left(\zeta=y\beta_y^{-1/2}\right)$ we write

$$x'=\frac{dx}{ds}=\beta_x^{1/2}\frac{d\eta}{ds}=\beta_x^{1/2}\frac{d\eta}{d\phi}\frac{d\phi}{ds}, \tag{20}$$

or, $x'=\beta_x^{1/2}\dot{\eta}\frac{1}{R}$. We note that $\dot{\eta}=\frac{d\eta}{d\phi}$, $\eta'=\frac{d\eta}{ds}$ and S=Rϕ so that $\frac{d\phi}{ds}=\frac{1}{\beta_x Q_x}=\frac{1}{R}$. Substituting the above relations in Eq. (1)

$$\varepsilon_x=\frac{1}{\beta_x}\left(\eta\beta_x^{1/2}\right)^2+\beta_x\left(\beta_x^{1/2}\frac{\dot{\eta}}{R}\right)^2, \tag{21}$$

or $\varepsilon_x=\eta^2+\frac{\beta_x^2}{R^2}\dot{\eta}^2$ and since $\beta_x=\frac{R}{Q_x}$, therefore we obtain

$$\varepsilon_x=\eta^2+\frac{\dot{\eta}^2}{Q_x^2}. \tag{22}$$

LINEARING COUPLING

In order to study the effect of field errors we should introduce a realistic field and add an additional term on the write hand side of Eqs .(2) and (12)

$$x''+Kx=\frac{\ddot{\eta}+Q_x^2\eta}{Q_x^2\beta_x^{3/2}}=\frac{\Delta B_y}{RB_0}. \tag{23}$$

In the presence of linear coupling, we define the average normalized skew gradient as

$K_0=\frac{1}{RB_0}\left(\frac{\Delta B_x}{\Delta x}\right)=\frac{1}{RB_0}\left(\frac{\Delta B_y}{\Delta y}\right)$, with B_x and B_y the normalized magnetic fields. From Eq. (23) the equations of betatron motion of a test particle i in the presence of linear coupling due to skew quadrupoles are given by [3]

$$\frac{d^2\eta_i}{d\phi_i^2}+Q_{x,i}^2\eta_i=\beta_{x,i}^{3/2}Q_{x,i}^2\frac{\Delta B_y}{RB_0}. \tag{24}$$

$$\frac{d^2\zeta_i}{d\phi_i^2}+Q_{y,i}^2\zeta_i=\beta_{y,i}^{3/2}Q_{y,i}^2\frac{\Delta B_x}{RB_0}. \tag{25}$$

Now, using the definition of skew gradient and also Eq. (5) we can write

$$\frac{d^2\eta_i}{d\phi_i^2}+Q_{x,i}^2\eta_i=\beta_{x,i}^{3/2}\beta_{y,i}^{1/2}Q_{x,i}^2K_i\zeta_i. \tag{26}$$

and a similar relation for y direction

$$\frac{d^2\zeta_i}{d\phi_i^2}+Q_{y,i}^2\zeta_i=\beta_{y,i}^{3/2}\beta_{x,i}^{1/2}Q_{y,i}^2K_i\eta_i. \tag{27}$$

Here, the normalized (Courant-Snyder) coordinates and angle are used

$$\eta_i=x_i\beta_{x,i}^{-1/2}(s),\zeta_i=y_i\beta_{y,i}^{-1/2}(s), \tag{28}$$

$$\phi_i=Q_{x,i}^{-1}\int_0^s\beta_{x,i}^{-1}(t)dt\approx Q_{y,i}^{-1}\int_0^s\beta_{y,i}^{-1}(t)dt, \tag{29}$$

where x_i and y_i are the horizontal and vertical deviations from the central orbit, s is the azimuthal coordinate and $\beta_{x,y}$ the betatron functions. Considering that the tunes and the skew gradient are the same for all the particles, i.e. $Q_{x,i}=Q_x$, $Q_{y,i}=Q_y$ and $K_i=K_0$, and using the smooth approximation $\beta_{x,i}=\beta_x\approx R/Q_x$ and $\beta_{y,i}=\beta_y\approx R/Q_y$ the equation of motion can be re-written as [3]

$$\frac{d^2\eta}{d\phi^2}+Q_x^2\eta=R^2\left(\frac{Q_x}{Q_y}\right)^{1/2}K_0\zeta, \tag{30}$$

$$\frac{d^2\zeta}{d\phi^2}+Q_y^2\zeta=R^2\left(\frac{Q_y}{Qx}\right)^{1/2}K_0\eta. \tag{31}$$

SUM OF EMITTANCES

Multiplying Eqs. (30) and (31), on both sides by $\frac{d\eta}{d\phi}$ and $\frac{d\zeta}{d\phi}$, respectively, we have

$$\frac{d\eta}{d\phi}\frac{d^2\eta}{d\phi^2}+Q_x^2\frac{d\eta}{d\phi}\eta=R^2\left(\frac{Q_x}{Q_y}\right)^{1/2}K_0\frac{d\eta}{d\phi}\zeta, \quad (32)$$

$$\frac{d\zeta}{d\phi}\frac{d^2\zeta}{d\phi^2}+Q_y^2\frac{d\zeta}{d\phi}\zeta=R^2\left(\frac{Q_y}{Qx}\right)^{1/2}K_0\frac{d\zeta}{d\phi}\eta \quad (33)$$

This can be re-written as

$$\frac{1}{2}\frac{d}{d\phi}\left(\frac{d\eta}{d\phi}\right)^2+\frac{1}{2}Q_x^2\frac{d\eta^2}{d\phi}=R^2\left(\frac{Q_x}{Q_y}\right)^{1/2}K_0\frac{d\eta}{d\phi}\zeta, \quad (34)$$

$$\frac{1}{2}\frac{d}{d\phi}\left(\frac{d\zeta}{d\phi}\right)^2+\frac{1}{2}Q_y^2\frac{d\zeta^2}{d\phi}=R^2\left(\frac{Q_y}{Qx}\right)^{1/2}K_0\frac{d\zeta}{d\phi}\eta. \quad (35)$$

or

$$\frac{d}{d\phi}\left(\frac{\eta'^2}{Q_x^2}+\eta^2\right)=\frac{2R^2}{Q_x^2}\left(\frac{Q_x}{Q_y}\right)^{1/2}K_0\frac{d\eta}{d\phi}\zeta, \quad (36)$$

$$\frac{d}{d\phi}\left(\frac{\zeta'^2}{Q_y^2}+\zeta^2\right)=\frac{2R^2}{Q_y^2}\left(\frac{Q_y}{Qx}\right)^{1/2}K_0\frac{d\zeta}{d\phi}\eta, \quad (37)$$

and using the definition of emittances, we can write

$$\frac{d}{d\phi}(\varepsilon_x+\varepsilon_y)=2R^2K_0\left(\sqrt{\frac{1}{Q_x^3Q_y}}\zeta\frac{d\eta}{d\phi}+\sqrt{\frac{1}{Q_y^3Q_x}}\eta\frac{d\zeta}{d\phi}\right). \quad (38)$$

For weak skew coefficient the right hand side can be neglected and therefore the sum of emittances remains constant. For the special case $Q_x=Q_y$ we can write

$$\frac{d}{d\phi}(\varepsilon_x+\varepsilon_y)=\frac{2R^2K_0}{Q_x^2}\left(\zeta\frac{d\eta}{d\phi}+\eta\frac{d\zeta}{d\phi}\right), \quad (39)$$

Therefore

$$\varepsilon_x+\varepsilon_y=\frac{2R^2K_0}{Q_x^2}\eta\zeta+\text{const.} \quad (40)$$

This relation clearly shows the effect of coupling η, ζ on the sum of emittances. In the case of weak skew quadruple coefficient, the first term on the right hand side is negligible and we can approximately reach a sum invariance

$$\varepsilon_x+\varepsilon_y=\text{const.} \quad (41)$$

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