AN EFFICIENT FORMALISM FOR SIMULATING THE LONGITUDINAL KICK FROM COHERENT SYNCHROTRON RADIATION*

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Abstract

Coherent Synchrotron Radiation (CSR) can severely limit the performance of planned light sources and storage rings which push the envelope to ever higher bunch densities. In order to better simulate CSR, the formalism of Saldin [E. L. Saldin, E. A. Schneidmiller, and M. V. Yurkov, Nucl. Instrum. Methods Phys. Res., Sect. A 398, 373 (1997)] is extended to work at lower energies and shorter length scales. The formalism is also generalized to cover the case of an arbitrary configuration of multiple bends.

INTRODUCTION

It is envisioned that future accelerators will call for shorter, higher intensity beams. A possible limiting factor in these efforts is the increased energy spread due to Coherent Synchrotron Radiation (CSR). In order to better simulate these effects, this paper extends the formalism of Saldin[1] to work at lower energies and shorter length scales. The formalism developed here is also generalized to cover the case of an arbitrary lattice configuration of bends and drifts.

TWO PARTICLE INTERACTION

The analysis starts by considering two particles of charge e following the same trajectory as shown in Figure 1. The Lienard–Wiechert formula gives the electric field $E(P)$ at the position of the kicked particle at point P and time t due to the source particle at point P' and retarded time t'

$$\mathbf{E}(\mathbf{P}) = \frac{e}{\gamma^2} \frac{\boldsymbol{L} - L\beta\boldsymbol{n}'}{(L - \boldsymbol{L}\cdot\beta\boldsymbol{n}')^3} + \frac{e}{c^2} \frac{\boldsymbol{L} \times [(\boldsymbol{L} - L\beta\boldsymbol{n}') \times \boldsymbol{a}']}{(L - \boldsymbol{L}\cdot\beta\boldsymbol{n}')^3} \tag{1}$$

It will be assumed that both particles have the same speed $\beta = v/c$, and $\boldsymbol{n}'$ and $\boldsymbol{n}$ are the unit velocity vectors for the source and kicked particles respectively. In Eq. (1), $\boldsymbol{L}$ is the vector from P' to P. The retarded time t' is related to t via $t - t' = L/c$. At time t, the source particle has a longitudinal position s' and the longitudinal position of the kicked particle at P is s. The distance $z \equiv s - s'$ between the particles at constant time is

$$z = L_s - \beta L \tag{2}$$

where L_s is the path length from P' to P. Generally, the relativistic approximation $\beta = 1$ will be made. However, some terms in $1 - \beta \simeq 1/2\,\gamma^2$ will need to be retained.

The first term on the right hand side of Eq. (1) has a $1/z^2$ singularity at small distances. Following Saldin[1], this singularity is dealt with by dividing the electric field into two parts. The space charge component $\boldsymbol{E}_{\text{SC}}$, which contains the singularity, is the field that would result if the particles where moving without acceleration along a straight line. The CSR term, $\boldsymbol{E}_{\text{CSR}}$, is what is left after subtracting off the space charge term

$$\boldsymbol{E}_{\text{SC}} \equiv \frac{e\,\boldsymbol{n}}{\gamma^2\,z^2}\,, \qquad \boldsymbol{E}_{\text{CSR}} \equiv \boldsymbol{E} - \boldsymbol{E}_{\text{SC}} \tag{3}$$

The rate $K \equiv d\mathcal{E}/ds$ at which the kicked particle is changing energy due to the field of the source particle is

$$K \equiv K_{\text{CSR}} + K_{\text{SC}} = e\,\boldsymbol{n}\cdot\boldsymbol{E}_{\text{CSR}} + e\,\boldsymbol{n}\cdot\boldsymbol{E}_{\text{SC}} \tag{4}$$

Following Saldin, the transverse extent of the beam will be ignored in the calculation of K_{CSR}. However, the inclusion of the finite beam size will be needed to remove the singularity in the calculation of K_{SC}.

CSR CALCULATION

The source point P' and the kick point P will, in general, not be within the same lattice element. Since the transverse extent of the beam is being ignored, all elements will be considered to be either bends or drifts.

In Figure 1, R is the bending radius and $g = 1/R$ is the bending strength of the element that contains the source point P'. The magnitude of the acceleration is $a' \simeq c^2/R$. This element ends at point $\mathcal{O}$. ϕ is the angle and $d = R\,\phi$ is the path length between P' and $\mathcal{O}$.

Between point $\mathcal{O}$ and the kick point P, d_i is the path length of the i^{th} element, $i = 1, \ldots, N-1$, where N is the number of elements in this region. For the last element d_N is the distance from the start of the element to point P. ϕ_i is the bend angle, R_i is the bend radius, and $g_i = 1/R_i$ is the bend strength for the i^{th} element. For a drift $\phi_i = g_i = 0$.

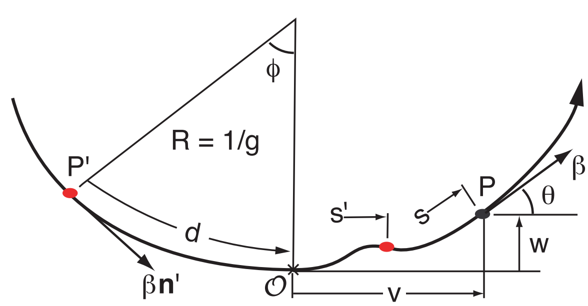

Figure 1: A particle at point P' kicks a particle at point P.

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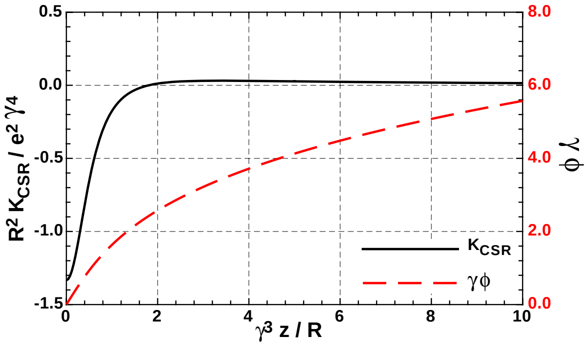

Figure 2: K_{CSR} and ϕ as a function of z for a bend.

In Figure 1, (v, w) are the coordinates of point P with respect to point $\mathcal{O}$ with the v–axis parallel to the orbit's longitudinal $\hat{\boldsymbol{s}}$ axis at point $\mathcal{O}$ and the w–axis pointing upwards towards the inside of the element containing the point P'. With the assumption that all bend angles are small, v and w can be approximated by

$$v = \nu_1 - \nu_3\,, \qquad \text{and} \qquad w = \omega_2 \tag{5}$$

where

$$\begin{aligned} \nu_1 &= \sum_{i=1}^{N} d_i\,, \qquad \omega_2 = \sum_{i=1}^{N} d_i \left(\psi_i + \frac{1}{2} g_i\, d_i\right) \\ \nu_3 &= \sum_{i=1}^{N} d_i \left(\frac{1}{2}\psi_i^2 + \frac{1}{2}\psi_i\, g_i\, d_i + \frac{1}{6} g_i^2\, d_i^2\right) \end{aligned} \tag{6}$$

where ψ_i is the orientation angle at the entrance end of the i^{th} element: $\psi_i = \sum_{k=1}^{i-1} \phi_k$. The above formulas are able to handle negative bends (beam rotating clockwise). For a negative bend R_i, g_i and ϕ_i are negative while $d_i = R_i\,\phi_i$ is always positive. The angle θ of the vector $\boldsymbol{n}$ with respect to the v–axis is $\theta = \sum_{i=1}^{N} g_i\, d_i$. In terms of v and w, the components of the vector $\boldsymbol{L}$ are

$$\begin{aligned} L_v &= [\nu_1 + d] - \left[\nu_3 + \frac{g^2\, d^3}{6}\right]\,, \qquad L_w = \omega_2 - \frac{g\, d^2}{2} \\ L &= [\nu_1 + d] - \left[\nu_3 + \frac{g^2\, d^3}{6} - \frac{1}{8}\frac{(2\,\omega_2 - g\, d^2)^2}{\nu_1 + d}\right] \end{aligned} \tag{7}$$

The path length is $L_s = d + \nu_1$. With Eq. (2) this gives

$$z = \frac{\nu_1 + d}{2\,\gamma^2} + \left[\nu_3 + \frac{g^2\, d^3}{6} - \frac{1}{8}\frac{(2\,\omega_2 - g\, d^2)^2}{\nu_1 + d}\right] \tag{8}$$

Substituting the above equations in Eq. (1), and (3) gives

$$\begin{aligned} K_{\text{CSR}} = 4\, e^2\, \gamma^4\, \tau^2 \Bigg\{ & \frac{g\,(\tau^2 - \alpha^2)\,(\alpha - \tau\,\kappa)}{(\tau^2 + \alpha^2)^3} + \\ & \frac{\tau^2 - \alpha^2 + 2\,\tau\,\alpha\,\kappa}{(\tau^2 + \alpha^2)^3} \Bigg\} - \frac{e^2}{\gamma^2\, z^2} \end{aligned} \tag{9}$$

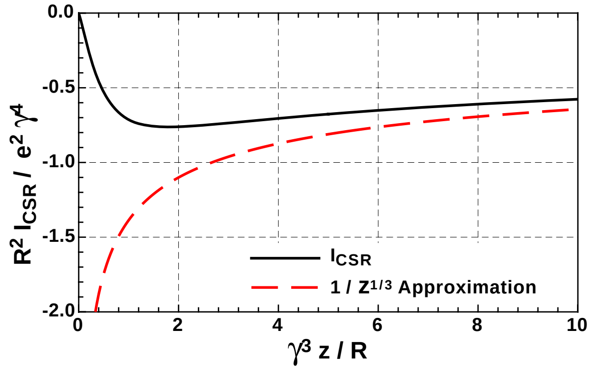

Figure 3: I_{CSR} a function of z for a bend. The dashed line is the large z approximation as given in Eq. (16).

where

$$\begin{aligned} \alpha &= \gamma^2 \left(\omega_2 + g\, d\, \nu_1 + \frac{1}{2}\, g\, d^2\right) \\ \kappa &= \gamma\,(\theta + g\, d)\,, \qquad \tau = \gamma\,(d + \nu_1) \end{aligned} \tag{10}$$

In the special case where points P and P' are within the same bend, Eq. (9) reduces to (cf. Saldin[1] Eq. (32))

$$\begin{aligned} K_{\text{CSR}} = \frac{4\, e^2\, \gamma^4}{R^2} \Bigg\{ & \frac{\hat{\phi}^2/4 - 1}{2\,(1 + \hat{\phi}^2/4)^3} + \\ & \frac{1}{\hat{\phi}^2} \left[\frac{1 + 3\,\hat{\phi}^2/4}{(1 + \hat{\phi}^2/4)^3} - \frac{1}{(1 + \hat{\phi}^2/12)^2}\right] \Bigg\} \end{aligned} \tag{11}$$

where $\hat{\phi} \equiv \gamma\,\phi$. Eq. (11) is valid for $\hat{\phi} > 0$. For $\hat{\phi} < 0$, K_{CSR} is, to a very good approximation, zero.

In the limit of small z, K_{CSR} has a limiting value of

$$K_{\text{CSR}}(z) \simeq \frac{-4\, e^2\, \gamma^4}{3\, R^2} \qquad \text{for} \quad z \ll \frac{R}{\gamma^3} \tag{12}$$

At large values of z, z is cubic in ϕ so that $\hat{\phi} \simeq (24\, z/R)^{1/3}$. With this, Eq. (11) becomes

$$K_{\text{CSR}}(z) \simeq \frac{4\, e^2}{3^{3/2}\, R^{2/3}\, z^{4/3}} \qquad \text{for} \quad z \gg \frac{R}{\gamma^3} \tag{13}$$

$K_{\text{CSR}}(z)$ for a bend is plotted in Figure 2. K_{CSR} changes sign at $z = 1.8\, R/\gamma^3$. The long tail at $z > 1.8\, R/\gamma^3$ cannot be neglected since the integral $\int ds'\, K_{\text{CSR}}(s - s')$ is zero.

The fact that K_{CSR} is highly peaked in amplitude near $z = 0$ can be problematic for simulations at ultra–relativistic energies since the characteristic longitudinal distance between particles or mesh points needs to be less than R/γ^3. One way of dealing with the peaked nature of K_{CSR} is to first consider the kick from a line of particles of density $\lambda(s)$ and then to integrate by parts

$$\begin{aligned} \left(\frac{d\mathcal{E}}{ds}\right)_{\text{CSR}} &= \int_{-\infty}^{\infty} ds'\, \lambda(s')\, K_{\text{CSR}}(s - s') \\ &= \int_{-\infty}^{\infty} ds'\, \frac{d\lambda(s')}{ds'}\, I_{\text{CSR}}(s - s') \end{aligned} \tag{14}$$

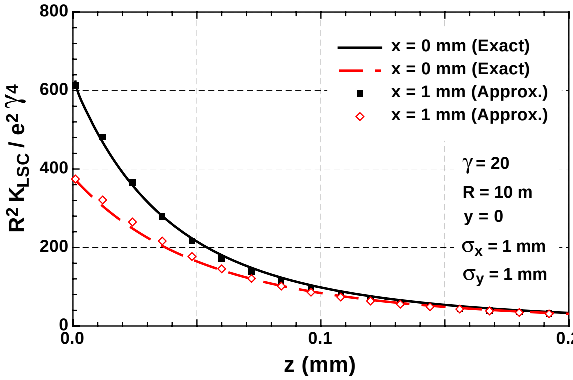

Figure 4: Comparison between Eq. (20) and an exact integration of Eq. (18).

where

$$I_{\text{CSR}}(s-s') = -\int_{-\infty}^{s'} ds''\, K_{\text{CSR}}(s-s'') \tag{15}$$

I_{CSR} is plotted in Figure 3 for a bend. The peaked nature of K_{CSR} has been smoothed over at the cost of having to deal with a derivative of λ. It is not possible to evaluate the integral of Eq. (15) in closed form. However, for $z \gg R/\gamma^3$, the approximation of Eq. (13) can be used with Eq. (11) to calculate an explicit ultrarelativistic equation for I_{CSR}[2]. For a bend this is

$$I_{\text{CSR}}(z) = \frac{-2\,e^2}{3^{1/3}\,R^{2/3}}\,\frac{1}{z^{1/3}} \qquad \text{for} \quad z \gg \frac{R}{\gamma^3} \tag{16}$$

Eq. (16) is plotted in Figure 3.

While, in general, it is helpful to have explicit formulas, for the purposes of evaluation within a simulation program this is not needed. The alternative is to use an exact implicit solution. Since Eq. (8) and Eq. (9) are rational functions, Eq. (15) can be integrated to give

$$I_{\text{CSR}}(s,s') = \frac{-2\,e^2\,\gamma\,(\tau+\alpha\,\kappa)}{\tau^2+\alpha^2} + \frac{e^2}{\gamma^2\,z} \tag{17}$$

Eq. (17) is the main result of this paper. Using Eq. (17), the integration of Eq. (14) in a simulation program can be done via interpolation of Eq. (8). Eq. (17) has several advantages over equations like Eq. (16). Eq. (17) is applicable at lower values of $\gamma^3 z$, That is, at lower energies and/or smaller length scales. Additionally, Eq. (17) has no singularity at small z, and it can be used to handle any combination of elements between the source and kick points.

SPACE CHARGE CALCULATION

The singularity at small z in the space charge term $\boldsymbol{E}_{\text{SC}}$ is removed by considering the finite transverse beam size. This term is equivalent to the problem of calculating the field given a static distribution of charges. It will be assumed that at any longitudinal position the transverse profile of the beam is Gaussian. Thus, a longitudinal slice of the beam will produce an energy change for a particle at at longitudinal offset z and transverse offset (x,y) from the slice center of

$$K_{\text{SC}}(z,x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'\,dy'\,\rho(x',y') \tag{18}$$
$$\frac{e^2\,\gamma\,z}{(\gamma^2\,z^2+(x-x')^2+(y-y')^2)^{3/2}}$$

where ρ is the bi-Gaussian distribution

$$\rho(x',y') = \frac{1}{2\,\pi\,\sigma_x\,\sigma_y}\,\exp\left[-\frac{x'^2}{2\,\sigma_x^2} - \frac{y'^2}{2\,\sigma_y^2}\right] \tag{19}$$

A heuristic solution for Eq. (18) in the region of interest ($x \lesssim 3\,\sigma_x$ and $y \lesssim 3\,\sigma_y$) is

$$K_{\text{SC}} = \frac{e^2\,\text{sgn}(z)}{\sigma_x\,\sigma_y\,\exp\left[\frac{x^2}{2\,\sigma_x^2}+\frac{y^2}{2\,\sigma_y^2}\right] + \frac{\sigma_x^2+\sigma_y^2}{\sigma_x+\sigma_y}\,\gamma\,z + \gamma^2\,z^2} \tag{20}$$

where

$$\text{sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \tag{21}$$

Eq. (20) is exact in the limit $z = 0$ and $z \to \infty$, and is an excellent approximation in the region in between. This is illustrated in Figure 4, which shows K_{SC} as a function of z as computed from an integration of Eq. (18) and from the approximate Eq. (20). The particular parameters chosen for the computation are given in the figure. Two cases were considered. One where the kicked particle is on-axis, and the other where the kicked particle is displaced $1\,\sigma_x$ off-axis. As can be seen, Eq. (20) gives an excellent approximation to the longitudinal space charge kick.

CONCLUSION

A general implicit formula for the longitudinal kick due to the coherent synchrotron radiation has been developed for particles on a common orbit. This formalism will handle any geometry of bends and drifts. For simulations, this formula is to be preferred over the explicit ultra–relativistic formula since the implicit formula does not have a singularity at $z = 0$ and is applicable at lower particle energies and smaller length scales.

Additionally, a heuristic formula for the longitudinal space charge kick has been presented which takes into account any transverse displacements of the kicked particles.

ACKNOWLEDGMENT

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REFERENCES

[1] E. L. Saldin, E. A. Schneidmiller and M. V. Yurkov, Nuc. Instrum. Methods Phys. Res. A, **398**, 373—394, (1997).

[2] P. Emma, and G.V. Stupakov, 2002 European Part. Acc. Conf., 1479, (2002).