Single Particle Linear and Nonlinear Dynamics

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Requirements from Accelerator Physics

- Small and large circular accelerators (radius of 1 to 4000 meters)
  - Low and high energy (100Mev to 7Tev), transition energy
  - Type of particles (electron, proton, and ions)
- Periodical system
  - S-coordinate as independent variable, resonances
  - Hill's equations, Courant-Synder parameters, and betatron tunes
- Intrinsic aberrations
  - Chromatic effects, compensation using sextupoles
  - Resonances driven by sextupoles, wiggler, space-charge, or beam-beam
- Practical aberrations
  - Misalignment of magnets, dynamical presentation of Euclidean group
  - Magnetic errors, harmonics, and fringe field
- Physics issues
  - Closed orbit, geometric and chromatic optics, modeling and correction
  - Symplectic condition, nonlinear resonances, and dynamic aperture

Approximated Presentations for Magnetic Elements in Codes

Lie factors

$$e^{:f_1:}e^{:f_2:}\ldots e^{:f_n:}$$

- engine in MARYLIE (A. Dragt)
- violates symplecticity when evaluates

Dragt-Finn

Taylor map

$$\mathcal{M}^n(z)$$

- engine in TRANSPORT, MAD, COSY (K. Brown and M. Berz), simple R-matrix
- but high-order one violates symplecticity

TPSA

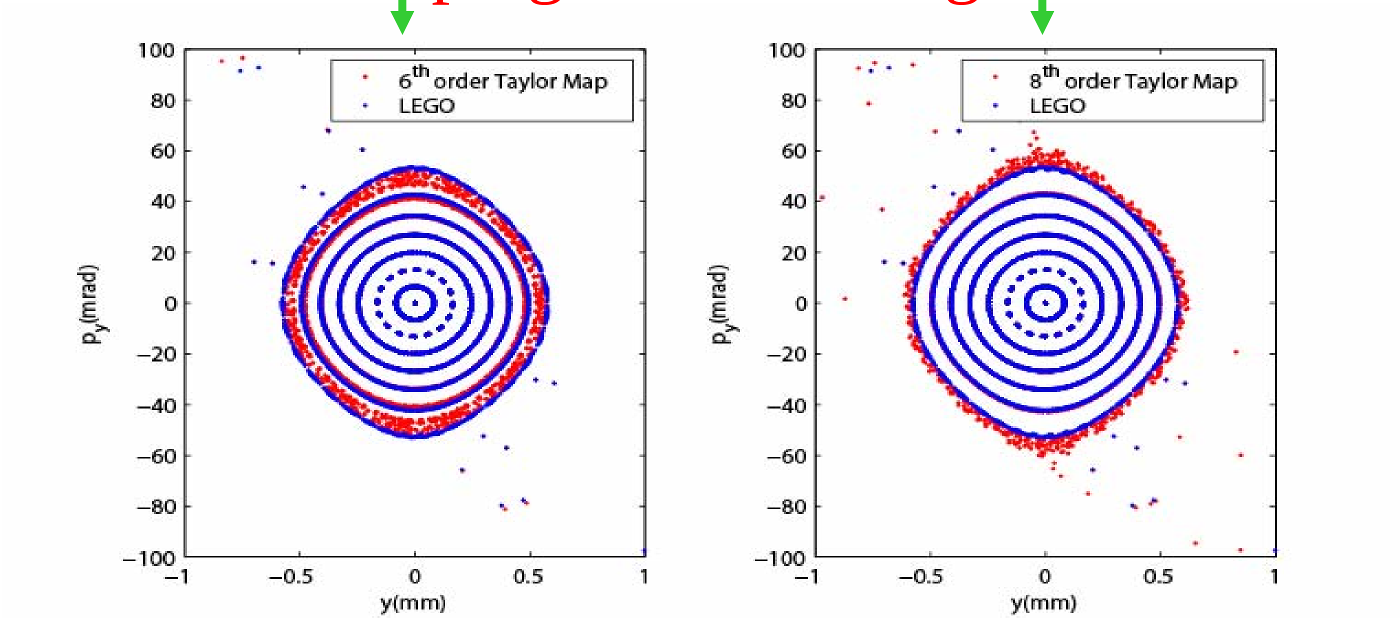
Canonical integrator

$$\prod_{i=1}^{n} e^{-\frac{:H_0:}{2}\Delta s} e^{-:H_1:\Delta s} e^{-\frac{:H_0:}{2}\Delta s}$$

- engine in TEAPOT, SAD, TRACY, LEGO, PTC (E. Forest, R. Ruth, and K. Hirata)
- preserves symplecticity
- simple and based on several known solutions
- emphasis on numerical process

Symplectic Conditions in Hamiltonian System

artificial damping or growth



$$M \circ J \circ M^T = J$$

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$ (one-dimension)

M is the Jacobian matrix of transfer map and J is symplectic unit matrix.

Hamiltonian for a Sector Bend with Mutipoles in Cylindrical Coordinate

TEAPOT&PTC&SAD:

solvable

$$H = -\frac{e}{cp_0}A_s(x,y) + \frac{x}{\rho} + \frac{x^2}{2\rho^2} - \left(1+\frac{x}{\rho}\right)\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}$$

(small ring)

paraxial approximation

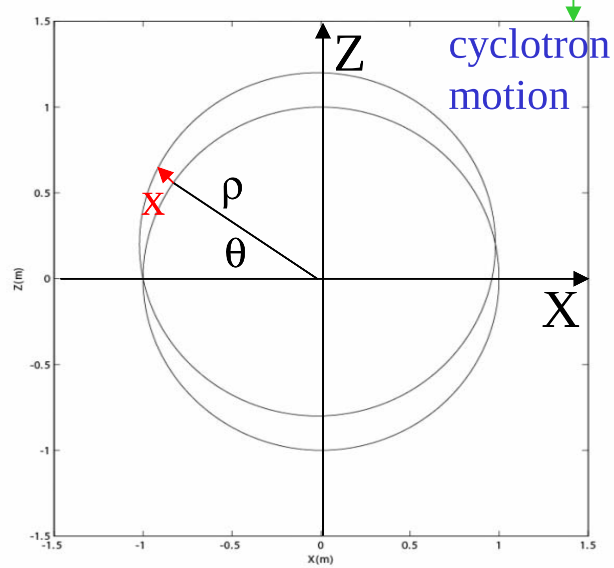
TRACY&DESPOT&LEGO&AT:

$$H = -\frac{e}{cp_0}A_s(x,y) + \frac{1}{2}\left(\frac{x}{\rho}\right)^2 - \frac{x}{\rho}\delta + \frac{p_x^2 + p_y^2}{2(1+\delta)}$$

(large ring)

where A_s is given by the harmonic expansion:

$$A_s(x,y) = -\mathrm{Re}\left(\sum_{n=1} \frac{1}{n}(b_n + ia_n)(x+iy)^n\right)$$



- $A_s(x,y)$ describes magnetic imperfection inside the body as well as many type of magnets such as quadrupole (b2)
- H can not be solved in general but can be split into solvable pieces

Second-Order Symplectic Integrators

Separate Hamiltonian into two exactly solvable parts:

$$H = H_0 + H_1$$

where $H_0 = \frac{x}{\rho} + \frac{x^2}{2\rho^2} - (1+\frac{x}{\rho})\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}$ and $H_1 = -\frac{e}{cp_0} A_s(x,y)$.

Approximation with symplectic integrators:

$$e^{-:H:L} = \prod_{i=1}^{n} e^{-:H:\Delta s} = \prod_{i=1}^{n} [e^{-\frac{:H_0:}{2}\Delta s} e^{-:H_1:\Delta s} e^{-\frac{:H_0:}{2}\Delta s} + O(\Delta s)^3]$$

exact propagate kick residual

or:

$$e^{-:H:L} = \prod_{i=1}^{n} e^{-:H:\Delta s} = \prod_{i=1}^{n} [e^{-\frac{:H_1:}{2}\Delta s} e^{-:H_0:\Delta s} e^{-\frac{:H_1:}{2}\Delta s} + O(\Delta s)^3]$$

- Symmetric and both are second-order integrator but the first one is commonly used
- Derived from the Baker-Cambell-Hausdorf formula
- Becomes the exact solution at the limit of infinite number of segments
- Preserves symplectic condition during the integration

“TEAPOT” as a Second-Order Integrators in Hamiltonian System

Separate H into three exactly solvable parts:

$$H_d = -\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}, \quad \longleftarrow \text{ drift}$$

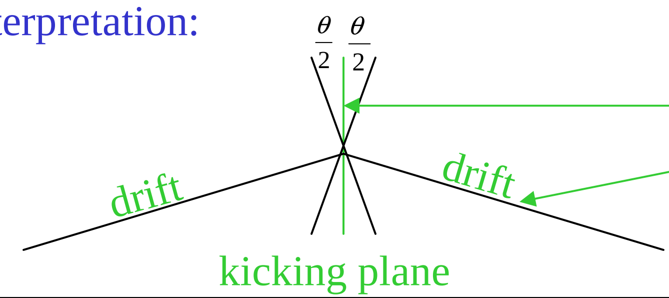
$$H_y = -\frac{x}{\rho}\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}, \quad \longleftarrow \text{ wedge (rotation in y axis)}$$

$$H_k = \frac{x}{\rho} + \frac{x^2}{2\rho^2} - \frac{e}{cp_0}A_s(x,y). \quad \longleftarrow \text{ kick}$$

Another type of second-order integrator:

$$e^{-:H:L} = \prod_{i=1}^{n}\left[e^{-\frac{:H_d:}{2}\Delta s} e^{-\frac{:H_y:}{2}\Delta s} e^{-:H_k:\Delta s} e^{-\frac{:H_y:}{2}\Delta s} e^{-\frac{:H_d:}{2}\Delta s} + O(\Delta s)^3\right]$$

- Not explicitly depend on the global coordinate and easy to extend to vertical bends
- Not unique and there are other symmetric combinations as well
- Has a geometric interpretation:



elements of Euclidean group

L. Shachinger and R. Talman, *Part. Accel.* **22**, 35 (1987)

High-Order Symplectic Integrators

Given a second-order integrator: $S_2(\Delta s)$, we can construct fourth-order integrator:

$$S_4(\Delta s) = S_2(x_1 \Delta s) S_2(x_0 \Delta s) S_2(x_1 \Delta s),$$

where $x_0 = -2^{\frac{1}{3}} / (2 - 2^{\frac{1}{3}}), x_1 = 1/(2 - 2^{\frac{1}{3}}).$

- For both the second and fourth order integrators, the result of integration always becomes exact at the limit that the number of segments approaches to infinity
- Fourth order integrator is not always more efficient than the second-order ones. It is more efficient to integrate quadrupole magnet. Usually, several segments are required.

R. Ruth, IEEE *Trans. Nucl. Sci. Ns*-**30**, 2669, (1983)

H. Yoshisa, *Phys. Lett.* A. **150**, 262, (1990).

Truncated Power Series Algebra

(M. Berz, *Part. Accel.* **24**, 109, 1989)

Consider a Taylor series:

$$f(z) = \sum_{k_1,k_2,k_3,k_4,k_5,k_6=0}^{k_1+k_2+k_3+k_4+k_5+k_6<order} D^f{}_{k_1,k_2,k_3,k_4,k_5,k_6}\, q_1^{k_1} p_1^{k_2} q_2^{k_3} p_2^{k_4} q_3^{k_5} p_3^{k_6}$$

Properties:

$$D^{f+g} = D^f + D^g$$ (derivatives of sum functions (f+g))

$$D^{\lambda f} = \lambda D^f$$ (derivatives of scales function λf)

$$D^{f \cdot g} = D^f \otimes D^g$$ (derivative of function fg)

defined as if the multiplication of two polynomials but truncated at the order

…

the n-th derivative: $$(f \cdot g)^{(n)}(x) = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} f^{(i)}(x) g^{(n-i)}(x)$$

“Polymorphic” Tracking in C++

Tracking phase vector

```
void Multpole::Kick(double l, double irho, Ray &
      v) const {
  double v1, v3, v5;
  double bx=0.0, by=0.0, byt=0.0;

  v1 = v(1);
  v3 = v(3);

  if ( order >= 1 ) {
    by = bn[order-1];
    bx = an[order-1];
   for (int j = order-1; j > 0; j--) {
      byt = v1 * by - v3 * bx + bn[j-1];
      bx  = v3 * by + v1 * bx + an[j-1];
      by  = byt;
      }
   }

   v(2) -= l * (by - (v5-v1*irho)*irho);
   v(4) += l * bx;
   v(6) += l * irho * v1;
}
```

Tracking a map

```
void Multpole::Kick(double l, double irho, Map
      & v) const {
  DA v1, v3, v5;
  DA bx=0.0, by=0.0, byt=0.0;

  v1 = v(1);
  v3 = v(3);

  if ( order >= 1 ) {
    by = bn[order-1];
    bx = an[order-1];
   for (int j = order-1; j > 0; j--) {
      byt = v1 * by - v3 * bx + bn[j-1];
      bx  = v3 * by + v1 * bx + an[j-1];
      by  = byt;
      }
   }

   v(2) -= l * (by - (v5-v1*irho)*irho);
   v(4) += l * bx;
   v(6) += l * irho * v1;
}
```

Courant-Snyder Parameters

One-turn matrix:

$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}$$

Rotation matrix:

$$R = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix}$$

We have:

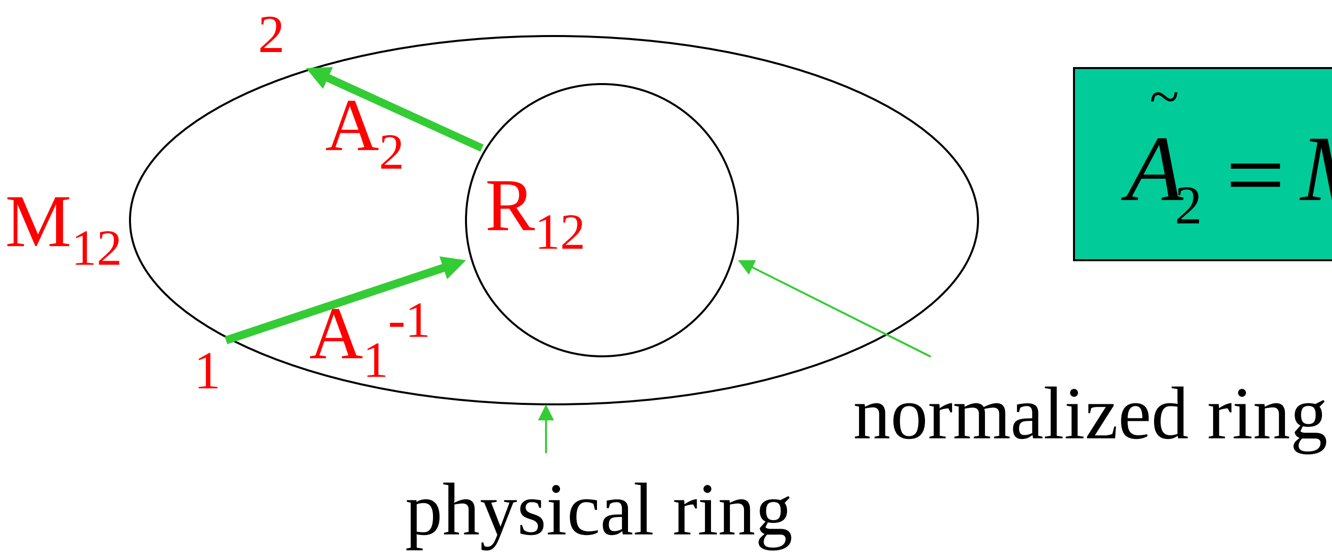
$$M = A \circ R \circ A^{-1}$$

where A^{-1} is a transformations from physical to normalized coordinates:

$$A^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}, A = \begin{pmatrix} \sqrt{\beta} & 0 \\ \frac{-\alpha}{\sqrt{\beta}} & \frac{1}{\sqrt{\beta}} \end{pmatrix}$$

However, the transformation is not unique: $\tilde{A} = A \circ R(\psi)$ will do the job as well.

Linear Lattice Propagation



$$\tilde{A}_2 = M_{12} \circ A_1$$

lattice functions at location 2:

$$\beta = \tilde{A}_{11}^2 + \tilde{A}_{12}^2, \alpha = -(\tilde{A}_{11}\tilde{A}_{21} + \tilde{A}_{12}\tilde{A}_{22}), \gamma = \tilde{A}_{21}^2 + \tilde{A}_{22}^2$$

phase advance: $\delta\psi_{12} = \tan^{-1}\tilde{A}_{12}/\tilde{A}_{11}$

Extending to coupled lattice: Y. Cai, *Phys. Rev. E.* ***68***, *036501 (2003)*

Nonlinear Normal Form

A nonlinear Taylor map can be normalized as well:

$$\mathcal{M}(J,\psi,\delta) = \mathcal{A}^{-1}(J,\psi,\delta)\circ\exp(-:\mathcal{H}(J,\delta):)\circ\mathcal{A}(J,\psi,\delta)$$

tune, chromaticity, detuning terms

and

$$\mathcal{A}(J,\psi,\delta)=\exp(:\mathcal{F}_N(J,\psi,\delta):)\ldots\exp(:\mathcal{F}_3(J,\psi,\delta):)\mathcal{A}_{,}(J,\psi)\mathcal{A}_{\eta}(\delta)$$

resonances lattice functions dispersion

where J and ψ are action-angle variables and δ is the relative momentum deviation.

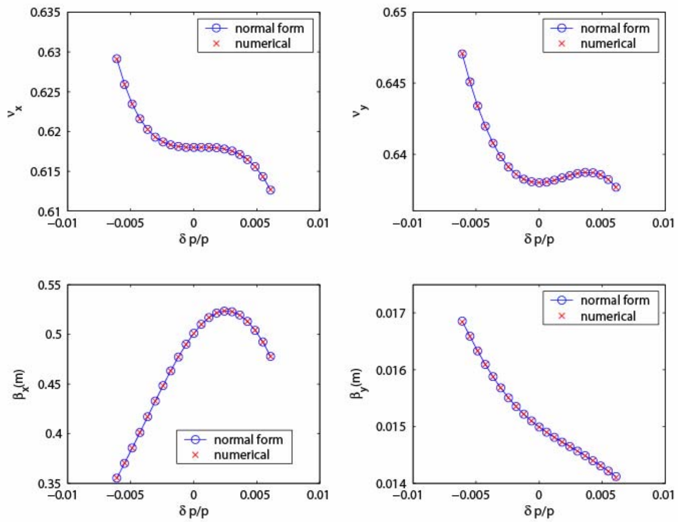
E. Forest, M. Berz, and J. Irwin, *P.A.* ***24***, *p91 (1989)*

Nonlinear Chromaticity in High Energy Ring of PEP-II

x y

ν(δ)

β*(δ)



$$\beta(\delta) = A_{11}(\delta)^2 + A_{12}(\delta)^2$$

obtained from analysis of nonlinear normal form

- Tow methods: normal form using LIELIB or numerical in LEGO agree well
- 10^{th} order Taylor map used for the normal form analysis
- Transformation from physical to normal coordinates becomes a function of $\delta = \Delta p/p$: $A^{-1}(\delta)$ but the same formula applies

High-Order Chromatic Optics Using Normal Form

$\nu_y(\delta)$

0	0.63799902339110182
1	-0.17158413929160805E-03
2	136.89736397776215
3	-21222.567788552446
4	-466110.23264129437
5	16474262.996298835
7	-18553799074.778057
8	7485639719025.4414

$\beta^*_y(\delta)$

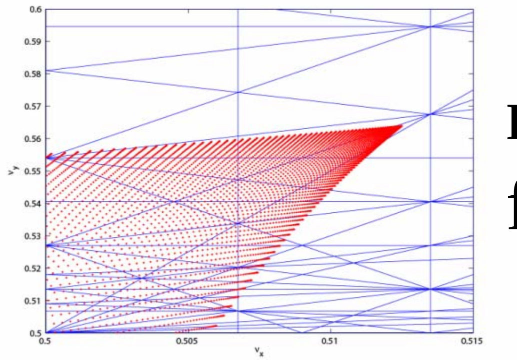
0	1.49929844883925460E-02
1	-1.63912178521324781E-01
2	1.42281439147605315E+01
3	-2.09846852191739163E+03
4	-1.30051472332955454E+04
5	1.27799196832761690E+07
6	1.53322278915590271E+07
7	-1.24725018999774094E+11
8	-9.77506834712234766E+12
9	3.31928901269866650E+15
10	-7.79515743334024300E+15

- A powerful extension to Harmon: arbitrary order of $\delta=\Delta p/p$ including x-y coupling
- Can have other parameter dependency, such as the strength of sextupoles
- Uniform analysis for both linear and nonlinear lattice. Easy to implemented
- Accurate high derivatives because of TPSA or DA technique

Tune Footprint for the Low Energy Ring (ν_x=38.5125, ν_y=36.5639)

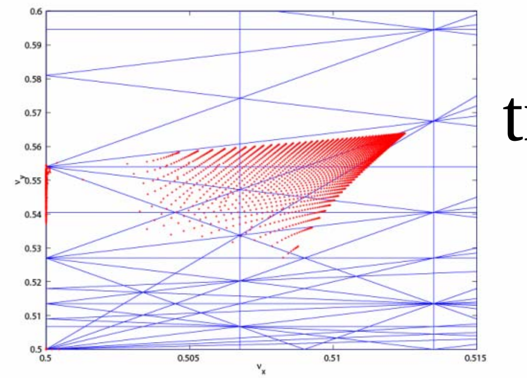
- Tune footprints calculated with normal form or with direct tracking agree well in the main stable region
- Resonance structure is not seen in the normal form approach because it tries to smooth out the resonances

using LEGO&LIELIB



normal form

using LEGO&NAFF



tracking

H.S. Dumas and J. Laskar, *Phys. Rev. Lett.* ***70***, *2975 (1993)*

Single Lie Presentation for One-Turn Map in Circular Accelerator

A single Lie presentation:

$$\mathcal{M}(J,\psi,\delta) = \mathcal{A}_1^{-1}(J,\psi,\delta)\circ\mathcal{R}(J)\exp(:f(J,\psi,\delta):)\circ\mathcal{A}_1(J,\psi,\delta)$$

linear normalization rotation

Nonlinear single Lie factor:

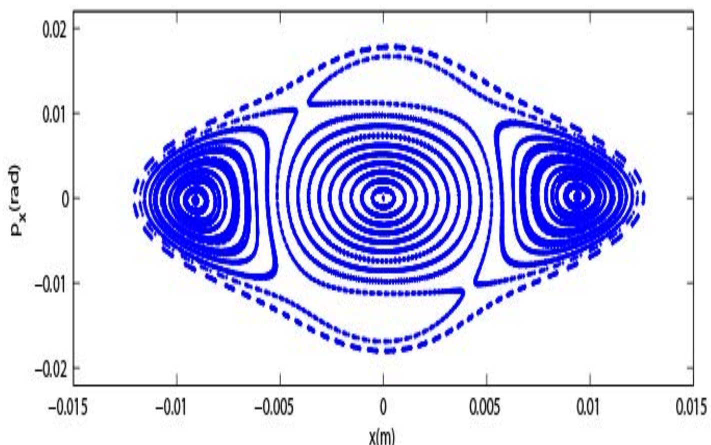
$$f(J,\psi,\delta) = \sum_{\vec{m},\vec{n},p}(2J_x)^{\frac{n_x}{2}}(2J_y)^{\frac{n_y}{2}}\delta^p[a_{\vec{m},\vec{n},p}\cos(m_x\psi_x + m_y\psi_y) + b_{\vec{m},\vec{n},p}\sin(m_x\psi_x + m_y\psi_y)]$$

- include both resonance driving terms and detuning terms ($m_x=m_y=0$)
- can be uses for tracking with multiple Poisson brackets (nPB)
- has been coded by Yiton Yan in C++: Zlib

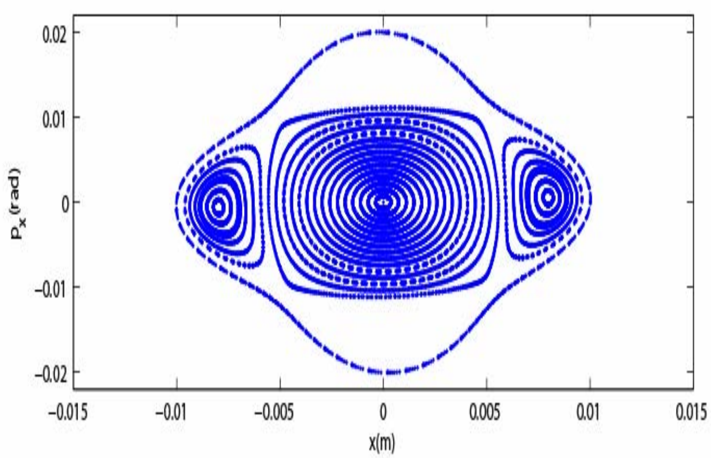
Y.T. Yan, J. Irwin, and T. Chen, *Part. Accel.* **55**, 263 (1996).

A 4th Order Resonance near Half Integer $4\nu_x$=154 in (LER)

element-by-element tracking (LEGO)



single-resonance map



- 10th order Taylor map is extracted from the design lattice of the Low Energy Ring in PEP-II
- 4th order resonance driving-term and a detuning term are taken out from the single Lie representation for a single turn
- Positions of fixed points and the width of resonance are computed analytically and iterations of the map is showed on the left
- Agreement between a simple perturbation theory and the direct tracking is about 15%

Smoothed and truncated Hamiltonian (ν=0.5125):

$$H = \nu J + \alpha(J) + f(J)\cos(m\psi - n\theta)$$

tune

detuning term

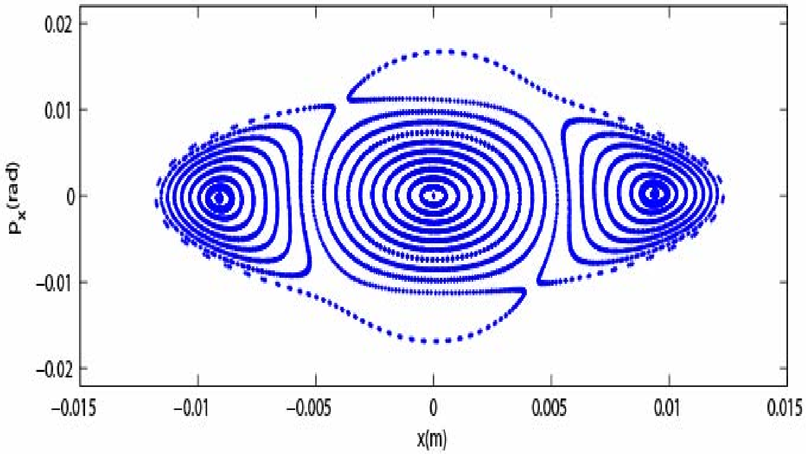
driving term

4

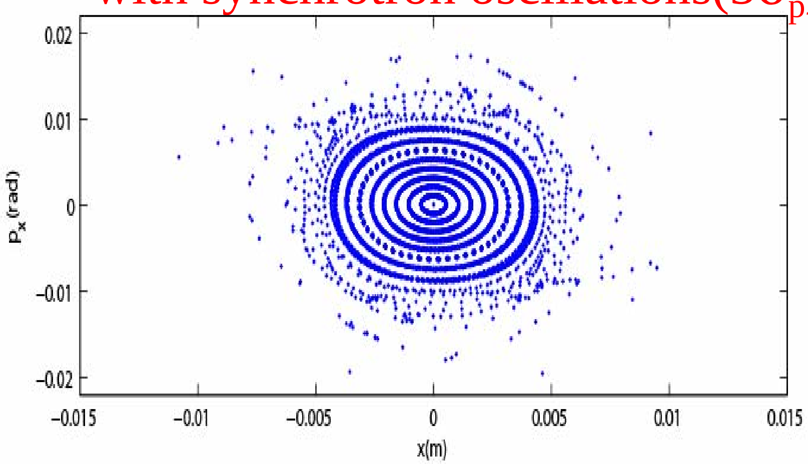
2

Dynamics Aperture of LER with Synchrotron Oscillations

without synchrotron oscillations

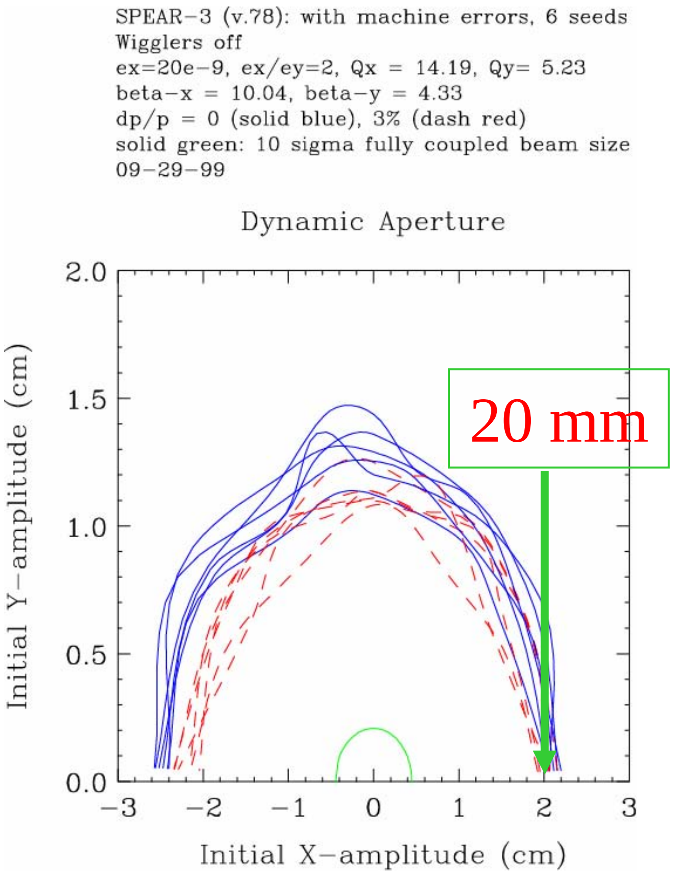


with synchrotron oscillations($5\sigma_p$)



Dynamics aperture with synchrotron oscillation is near the separatrix of the 4^{th} order resonance we have seen when the synchrotron oscillation is turned off. In this simple but realistic case, when a single and isolated resonance dominates, the dynamic aperture can be estimated using nonlinear transfer map and perturbation theory without tracking. Can we do better?

Dynamic Aperture of SPEAR3

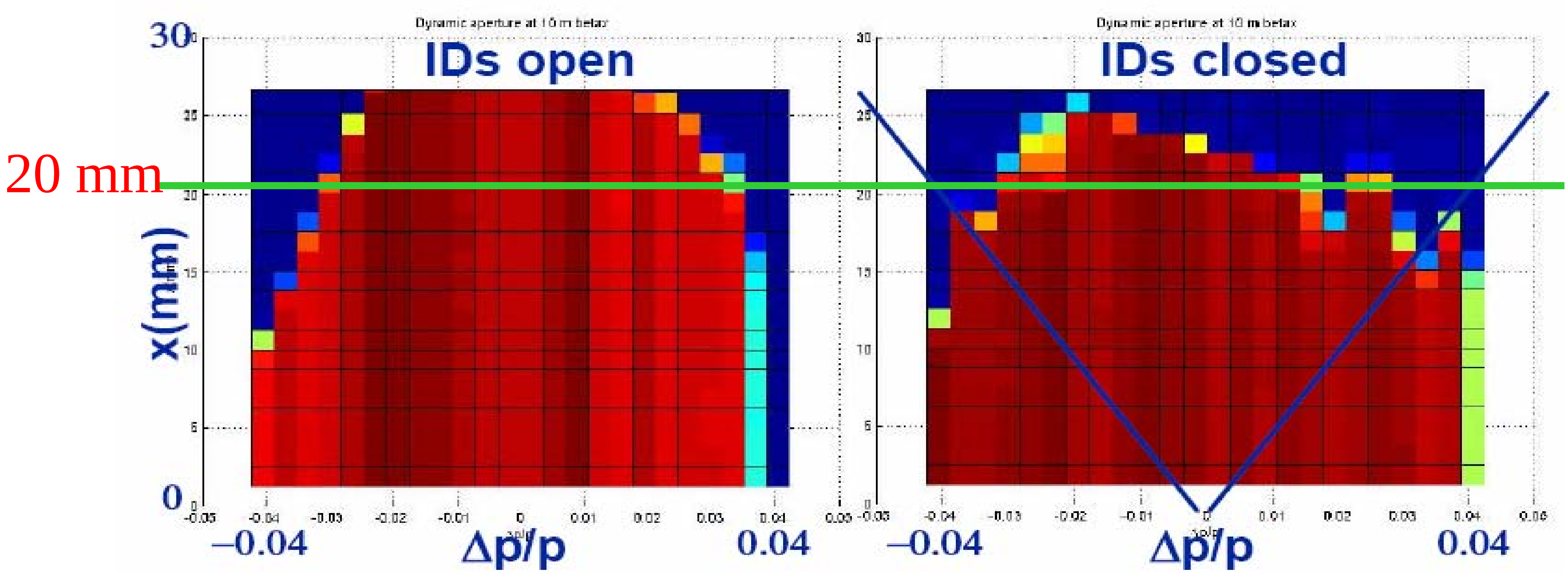


- Tracked with LEGO
- Misalignments
- Strength errors
- Multipole errors
- Orbit steering
- Coupling correction
- With synchrotron oscillation
- On and off momentum

Y. Nosochkov and J. Corbett, *SLAC-PUB-7965, 1998*

Measured Dynamic Aperture vs. $\Delta p/p$ at SPEAR3

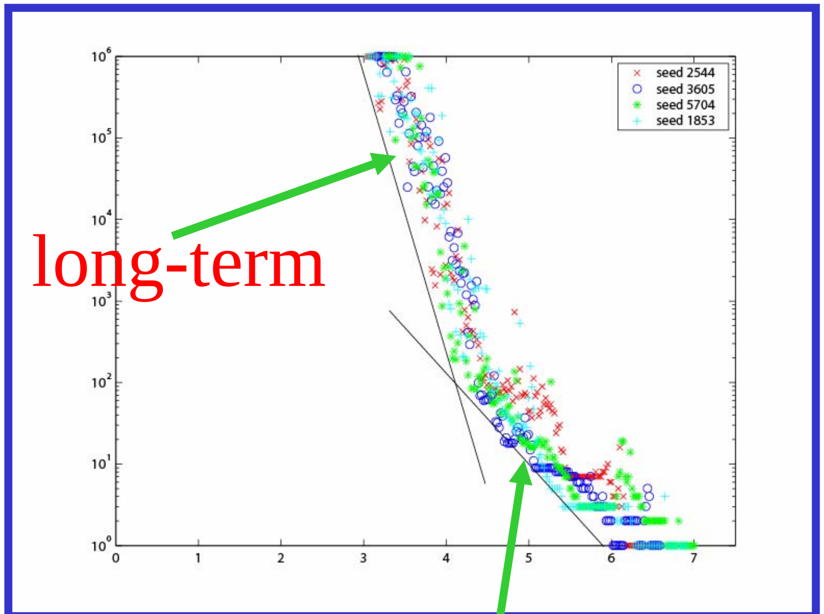
- Dynamic aperture measured with single injection kicker for varying rf frequency.



Courtesy of J. Safranek, *Theory club talk at SLAC, April, 2004*

Long-Term Dynamic Aperture

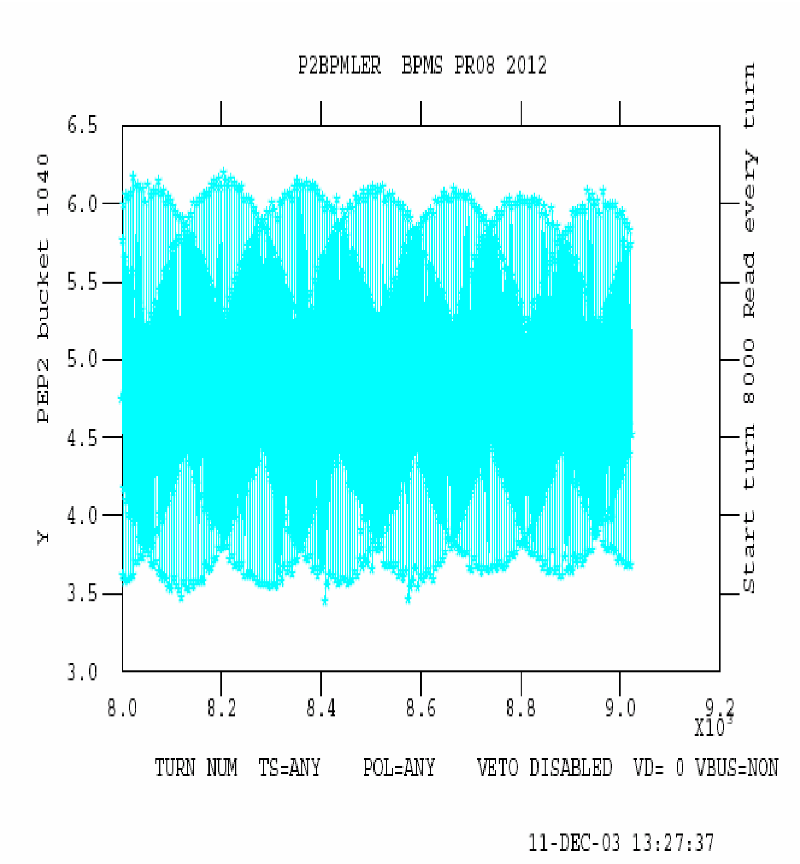
Survival plot for SSC



short-term

- A million turns which is about one tenth of the injection period is tracked
- All magnetic errors are included in the lattice
- Tracking with element-by-element using symplectic integrators in LEGO
- Synchrotron period defines the boundary between the short-term and long-term stability of the particle.

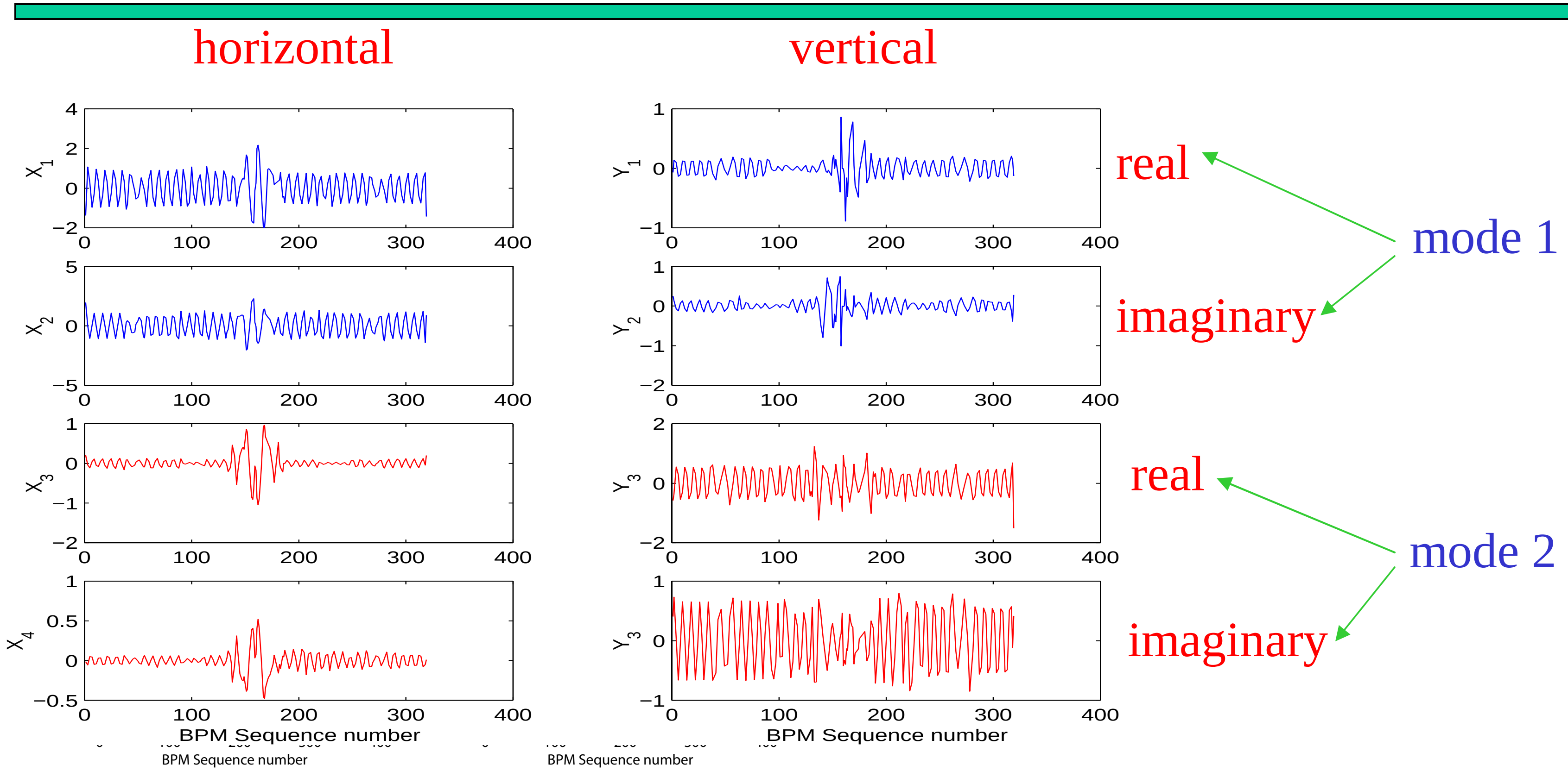
Coherently Excited Betatron Motion and Turn-by-Turn data



- Beam excited at eigen frequency in x or y
- Equilibrium reached due to radiation damping or decoherence
- Take turn-by-turn reading at all beam position monitors up to 1024 turns
- The phase advances between the beam position monitors can be accurately measured

J. Borer, C. Bovet, A. Burns, and G. Morpurgo, *Proc. The 3rd EPAC, p1082 (1992)*

In addition, Four Eigen Orbits Extracted Using FFT



• These orthogonal orbits are the Fourier transforms of the turn-by-turn readings of beam position monitors at the driving frequency. Since the peak in the spectrum can be located accurately, they can be measured precisely as well.

R-Matrix Elements Derived from Four Orthogonal Orbits

$$R_{12}^{ab} = (x_1^a x_2^b - x_2^a x_1^b)/Q_{12} + (x_3^a x_4^b - x_4^a x_3^b)/Q_{34}$$

$$R_{14}^{ab} = (y_1^a x_2^b - y_2^a x_1^b)/Q_{12} + (y_3^a x_4^b - y_4^a x_3^b)/Q_{34}$$

$$R_{32}^{ab} = (x_1^a y_2^b - x_2^a y_1^b)/Q_{12} + (x_3^a y_4^b - x_4^a y_3^b)/Q_{34}$$

$$R_{34}^{ab} = (y_1^a y_2^b - y_2^a y_1^b)/Q_{12} + (y_3^a y_4^b - y_4^a y_3^b)/Q_{34}$$

where *a* and *b* are indices for the locations of the beam position monitors, Q_{12} and Q_{34} are global invariance of the orbits. For general orbits, the relationship is much more complicated.

BPM Gains and Couplings

$$\begin{pmatrix} \mathcal{R}_{12}^{ab} \\ \mathcal{R}_{14}^{ab} \\ \mathcal{R}_{32}^{ab} \\ \mathcal{R}_{34}^{ab} \end{pmatrix} = \begin{pmatrix} g_x^b g_x^a & g_x^b \theta_{xy}^a & \theta_{xy}^b g_x^a & \theta_{xy}^b \theta_{xy}^a \\ g_x^b \theta_{yx}^a & g_x^b g_y^a & \theta_{xy}^b \theta_{yx}^a & \theta_{xy}^b g_y^a \\ \theta_{yx}^b g_x^a & \theta_{yx}^b \theta_{xy}^a & g_y^b g_x^a & g_y^b \theta_{xy}^a \\ \theta_{yx}^b \theta_{yx}^a & \theta_{yx}^b g_y^a & g_y^b \theta_{yx}^a & g_y^b g_y^a \end{pmatrix} \begin{pmatrix} R_{12}^{ab} \\ R_{14}^{ab} \\ R_{32}^{ab} \\ R_{34}^{ab} \end{pmatrix}$$

where g_x, g_y are gains and θ_{xy}, θ_{yx} are cross-coupling between x and y.

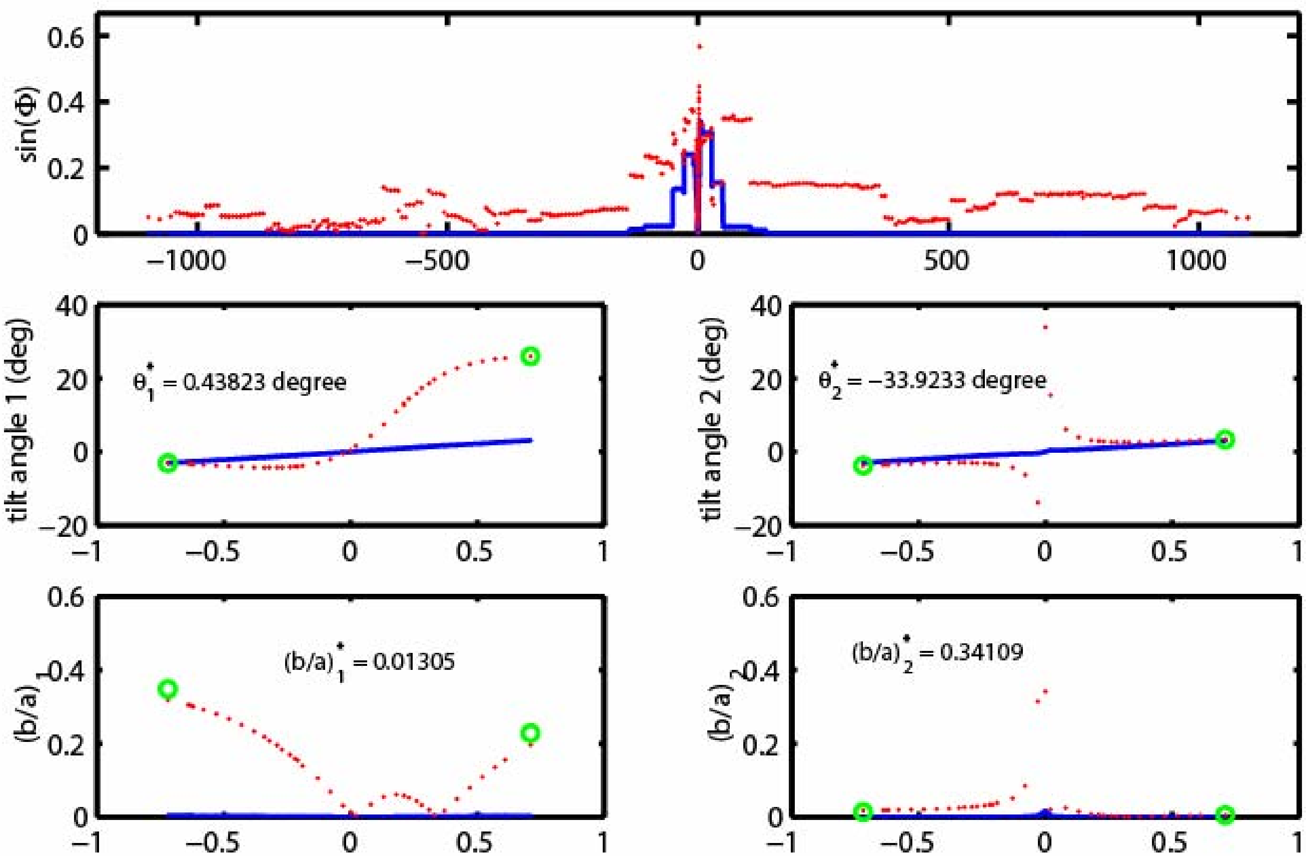
Measured

$$\mathcal{x} = g_x x + \theta_{xy} y$$

$$\mathcal{y} = g_y y + \theta_{yx} x$$

Beam

Local Coupling in the LER
(October 20, 2003)



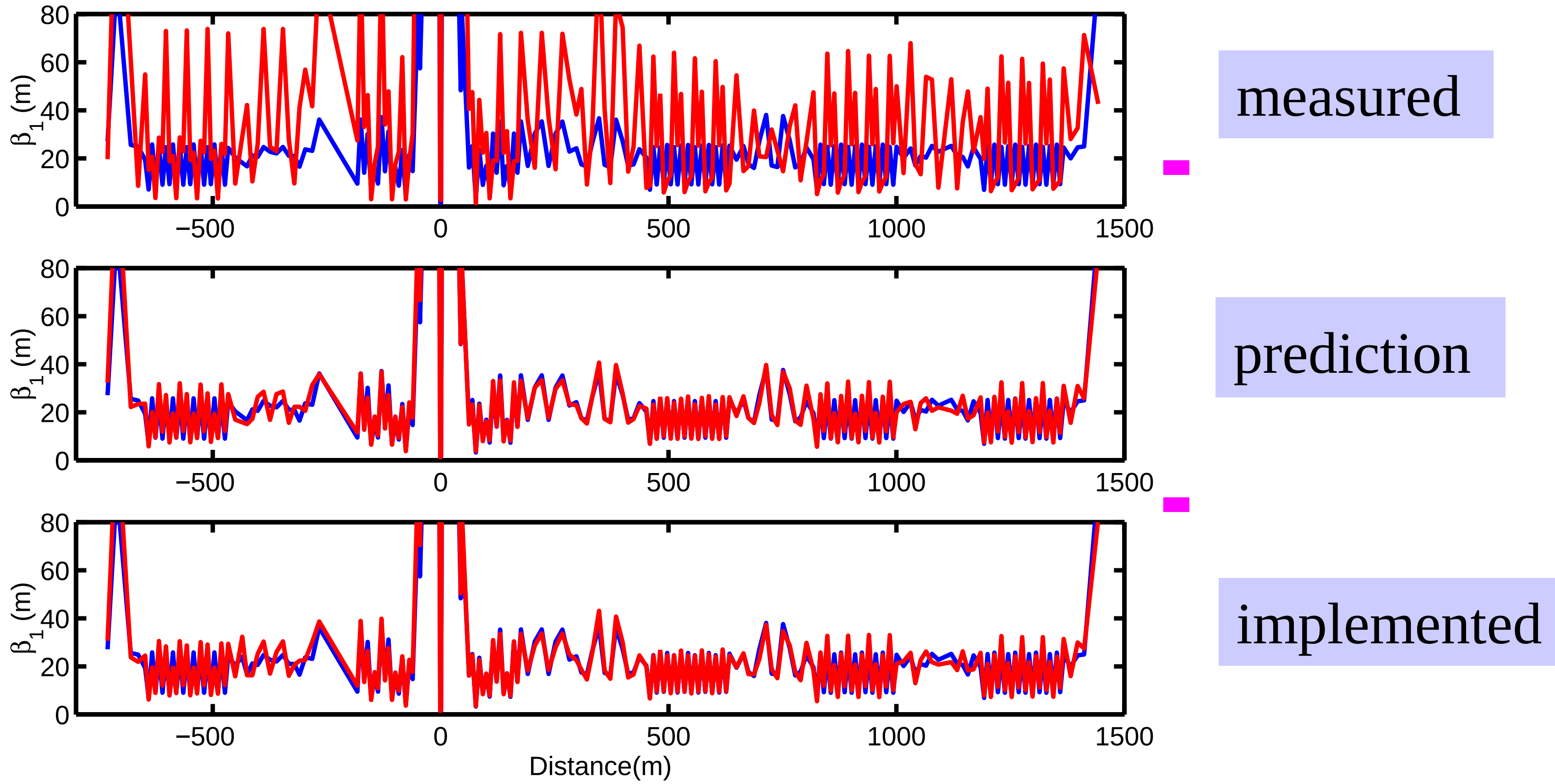
whole ring

near the interaction point

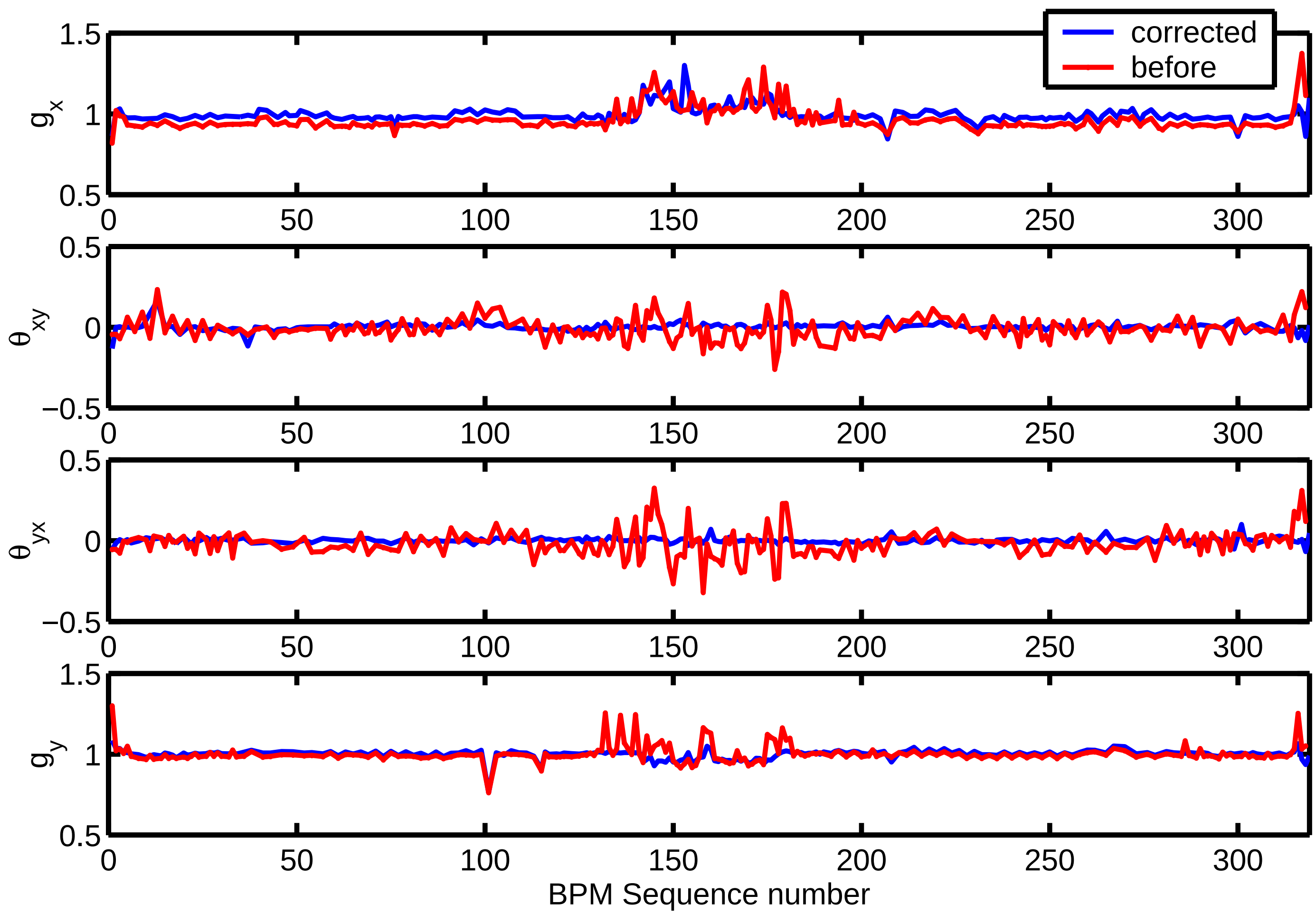
mode 1

mode 2

β Beating correction for the High Energy Ring (November 4, 2003)



Correction of Gains and coupling



X gain

Y to X

X to Y

Y gain

Summary and Conclusion

- Canonical integrators using together with the truncated power series provide us a powerful and yet simple scheme for the tracking and analysis of single particle in circular accelerators. It has intrinsically inherited consistency between the tracking and its very order-order map.
- TEAPOT can be considered as an application of symplectic integrators, which make it applicable to non-planar accelerators.
- Nonlinear normal form can be applied to compute the chromatic lattices order-by-order with respects to d to an arbitrary order. This is extension to HARMON.
- Dynamic aperture simulated by tracking is reliable within 20% compared with the experimental measurements provided the accurate inputs of machine errors.
- Smoothed and truncated Hamiltonian can be derived from the one-turn map and is useful way to understand the resonances in circular accelerators.
- Similar to the accurate tune measurement, optical measurement based on coherently excited orbits and frequency analysis has shown its advantages in accuracy and simplicity when it applies to highly coupled rings