ELECTROMAGNETIC FIELDS OF AN OFF-AXIS BUNCHED BEAM IN A CIRCULAR PIPE WITH FINITE CONDUCTIVITY AND THICKNESS - I

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Abstract

We compute the multipole expansion of the Green's function for an off-axis point particle running at constant velocity parallel to the axis of circular pipe with finite wall conductivity and thickness.

INTRODUCTION

Wake fields describe the interaction between a particle beam and the surrounding pipe wall. For perfectly conducting pipes and ultrarelativistic motion $(v = c)$ wakefields are negligibile. In the realistic case of walls of finite conductivity, and/or relatively low values of the relativistic factor γ, occurring, e.g., at injection, wake fields might be quite relevant. In addition, for low revolution frequencies, the finite thickness of the pipe wall should be properly taken into account [1]. Much has been written on the subject of wake fields, since the early work of Piwinski [2], who first studied the opposite limiting cases of a metal-coated ceramic vacuum chamber, where the coating is much thinner than the EM penetration depth, and of a homogeneous conducting pipe, much thicker than the EM penetration depth. Palumbo and Vaccaro extended Piwinsky's results for this latter case, by computing higher order wake-field multipoles [3]. Chao first gave a formula which fully exploits the dependence of the wakefield on the pipe wall thickness, but his analysis was restricted to the monopole term [4]. More recently, Ohmi and Zimmerman presented a thorough analysis of the subrelativistic effect [5]. Finally, Yokoya and Shobuda studied the finite-conductivity, finite-thickness pipe-wall problem, in the frame of a transmission line analogy, which can be applied to beam pipes with general transverse geometry and multi-layered walls, in the limit where the EM skin depth is much smaller than the (smallest) pipe transverse dimension [6].

In this communication we solve in full generality the problem of computing the wake field multipoles set up by an (offset) multi-bunch beam in a circular pipe with finite wall conductivity and thickness. The simplest circular geometry is considered. The exploited solution is exact but complicated, so that in most cases of practical interest one may resort to suitable limiting forms which are discussed in a companion paper.

THE WAKE FIELD

In this section we compute the Fourier transform of the wake potential Green's function produced by a point charge q_o running at a constant velocity $\beta c\hat{u}_z$, at an azimuthal coordinate θ_o and a distance r_o off axis of a circular cylindrical pipe with an inner radius b, conductivity σ, finite thickness Δ.

In order to compute the Green's function within the hollow pipe $(r < b)$, one has to write down the solution also in the conducting wall $(b \le r \le b+\Delta)$, and outside the beam liner $(r > b + \Delta)$, and enforce all needed boundary conditions. Here we limit ourselves to sketching the procedure and giving the final result. Full details will be published elsewhere.

The charge density of a (bunched) beam running parallel to the pipe axis, while preserving its longitudinal profile, can be written as the product of a transverse and longitudinal density

$$\rho(r, \theta, \xi) = \rho_t(r, \theta)\ f(\xi), \tag{1}$$

where r and θ are the radial and azimuthal coordinate respectively, and $\xi = z - \beta ct$. The related (scalar) potential Φ depends in turn on z and t only through ξ, and the wave equation for Φ accordingly reads:

$$\nabla_t^2\Phi + \frac{1}{\gamma^2}\frac{\partial^2\Phi}{\partial\xi^2} = -\frac{\rho(r, \theta, \xi)}{\epsilon_0}, \tag{2}$$

where as usual $\gamma = (1 - \beta^2)^{-1/2}$. Given the (running) point source

$$\delta(r, \theta, \xi|r_0, \theta_0, \xi_0) = \frac{\delta(r - r_o)}{r_o}\delta(\theta - \theta_0)\delta(\xi - \xi_0), \tag{3}$$

we shall seek the associated potential G (Green's function),

$$\nabla_t^2 G + \frac{1}{\gamma^2}\frac{\partial^2 G}{\partial\xi^2} = -\frac{\delta(r, \theta, \xi|r_0, \theta_0, \xi_0)}{\epsilon_0}. \tag{4}$$

The general solution of (2) can be written:

$$\Phi(r, \theta, \xi) = \int_0^{2\pi} r_o d\theta_o \int_0^b dr_0 \cdot$$

$$\int_{-\infty}^{\infty} d\xi_0\ \rho_t(r_0, \theta_0) f(\xi_0)\ G(r, \theta, \xi|r_0, \theta_0, \xi_0). \tag{5}$$

in view of the obvious representation

$$\rho(r,\theta,\xi) = \int_0^{2\pi} r_0 d\theta_0 \int_0^b dr_0 \cdot$$

$$\int_{-\infty}^{\infty} d\xi_0\ \rho_t(r_0,\theta_0) f(\xi_0)\delta(r, \theta, \xi|r_0, \theta_0, \xi_0), \tag{6}$$

and of the linearity of Eq.(2). The solution of (4) admits the following Fourier representation, where $\phi = \theta - \theta_0$ and $s = \xi - \xi_0$:

$$G(s, r, r_0, \phi) = \sum_{m=-\infty}^{\infty} G_m(s, r, r_0) e^{im\phi},$$

where:

$$G_m(s, r, r_0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{G}_m(k, r, r_0) e^{iks} dk. \quad (7)$$

Inserting Eq.s (7) into Eq.(5) we get:

$$\Phi(r, \phi, s) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im\phi} \left(\int_0^b r_0 dr_0 \, \rho_{t,m}(r_0) \right) \cdot$$

$$\int_{-\infty}^{\infty} \tilde{G}_m(k, r, r_0) F(k) \, e^{iks} dk \,, \quad (8)$$

where:

$$\rho_{t,m}(r_0) = \frac{1}{2\pi} \int_0^{2\pi} \rho_t(r_0, \theta_0) \, e^{im\theta_0} \, d\theta_0, \quad (9)$$

$$F(k) = \int_{-\infty}^{\infty} f(s) \, e^{-iks} \, ds, \quad (10)$$

are the (transverse) source azimuthal harmonic and the Fourier spectrum of the longitudinal source profile, respectively.

The unknowns $\tilde{G}_m(\cdot)$ in Eq.(8) are readily found. Using the obvious identities

$$\delta(s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iks} dk, \quad \delta(\phi) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im\phi}, \quad (11)$$

in Eq.s(3), (4) one readily gets an equation for $\tilde{G}_m(k, r, r_0)$:

$$\frac{d^2\tilde{G}_m}{dr^2} + \frac{1}{r}\frac{d\tilde{G}_m}{dr} - \left[\frac{m^2}{r^2} + \left(\frac{k}{\gamma}\right)^2\right]\tilde{G}_m = -\frac{1}{2\pi\epsilon_0}\frac{\delta(r-r_0)}{r_0}, \quad (12)$$

whose solution is a superposition of modified Bessel functions I_m and K_m, viz.:

$$\tilde{G}_m(k,r,r_0) = \frac{q_0}{2\pi\epsilon_0} \left\{ A(k, r, r_0) + B_m I_m\left(\frac{kr}{\gamma}\right) \right\}, \quad (13)$$

where

$$A(k, r, r_0) = \begin{pmatrix} K_m\left(\frac{kr}{\gamma}\right) I_m\left(\frac{kr_0}{\gamma}\right) \\ K_m\left(\frac{kr_0}{\gamma}\right) I_m\left(\frac{kr}{\gamma}\right) \end{pmatrix} \quad \begin{matrix} r_0 \le r \le b, \\ r \le r_0, \end{matrix} \quad (14)$$

and the constant B_m follows by enforcing suitable boundary conditions at $r = b$. For the special case of a perfectly conducting wall ($\sigma \longrightarrow \infty$), which will be henceforth identified with the ∞ superscript, one has:

$$\tilde{G}^{\infty}_m(k, r, r_0) = \frac{q_o}{2\pi\epsilon_o} \cdot$$

$$\left\{ A(k, r, r_0) - \frac{I_m\left(\frac{kr_o}{\gamma}\right)}{I_m\left(\frac{kb}{\gamma}\right)} K_m\left(\frac{kb}{\gamma}\right) I_m\left(\frac{kr}{\gamma}\right) \right\}, \quad (15)$$

For a pipe with finite wall conductivity and thickness, one has to write down the unknown Green's function, by solving suitable (homogeneous) equations also in the $b \le r \le d$ and $r \ge d$ regions, $d = b + \Delta$ being the external pipe radius, in order to write down the boundary conditions at $r = b$ and $r = b + \Delta$ needed to determine B_m. After some lenghty algebra we get the following solution describing the Green function for $r \le b$:

$$\tilde{G}_m(k, r, r_0) = \tilde{G}^{\infty}_m(k, r, r_0) +$$

$$\frac{q_o}{2\pi\epsilon_o} \frac{\gamma \, I_m\left(\frac{kr_o}{\gamma}\right) \;\; I_m\left(\frac{kr}{\gamma}\right)}{b \; k \; I_m\left(\frac{k \, b}{\gamma}\right)} \; \frac{N(k)}{D(k)}, \quad (16)$$

where:

$$N(k) = \bar{k}^2 K'_m(k'd) \left[I_m\left(\bar{k}b\right) K_m\left(\bar{k}d\right) - I_m\left(\bar{k}d\right) K_m\left(\bar{k}b\right)\right] +$$
$$+\eta \, \bar{k}k' K_m(k'd) \left[K_m\left(\bar{k}b\right) I'_m\left(\bar{k}d\right) - I_m\left(\bar{k}b\right) K'_m\left(\bar{k}d\right)\right], \quad (17)$$

$$D(k) = \bar{k}^2 I'_m(k'b) K'_m(k'd) \left[I_m\left(\bar{k}b\right) K_m\left(\bar{k}d\right) - I_m\left(\bar{k}d\right) K_m\left(\bar{k}b\right)\right]$$
$$+\eta \bar{k}k' I_m(k'b) K'_m(k'd) \left[K'_m\left(\bar{k}b\right) I_m\left(\bar{k}d\right) - I'_m\left(\bar{k}b\right) K_m\left(\bar{k}d\right)\right]$$
$$+\eta \bar{k}k' K_m(k'd) I'_m(k'b) \left[I'_m\left(\bar{k}d\right) K_m\left(\bar{k}b\right) - K'_m\left(\bar{k}d\right) I_m\left(\bar{k}b\right)\right]$$
$$+\eta^2 k'^2 I_m(k'b) K_m(k'd) \left[I'_m\left(\bar{k}b\right) K'_m\left(\bar{k}d\right) - I'_m\left(\bar{k}d\right) K'_m\left(\bar{k}b\right)\right] \quad (18)$$

with $k' = k/\gamma$ and

$$\eta = \frac{Z_o \sigma}{ik\beta} - 1. \quad (19)$$

It can be checked that Eq.(16) reduces to the solution obtained in [3] in the limit $d \to \infty$ of an infinitely thick wall.

BUNCHED BEAM SPECTRA

In storage rings and circular machines the beam is a periodic train of bunches, with spatial period $T_s = L_c/N_b$, where L_c is the ring circumference and N_b the total number of bunches. The function $f(\cdot)$ in Eq.(1) is thus:

$$f(s) \;=\; \sum_{n=-\infty}^{\infty} f_n e^{i2\pi(N_b/L_c)n \; s}, \quad (20)$$

where:

$$f_n = \frac{N_b}{L_c} \int_{[L_c/N_b]} f(s) e^{-i2\pi(N_b/L_c)n \; s} ds$$

$$\approx \frac{N_b}{L_c} F_1 \left(2\pi \frac{N_b}{L_c} n \right), \tag{21}$$

and F_1 is the Fourier transform of a *single* bunch with assumed typical length $\sigma_s \ll L_c/N_b$.
From Eq.s (10), (20) and (21) we get:

$$F(k) = \int_{-\infty}^{\infty} f(s) e^{-iks}\, ds \; =$$

$$= \; 2\pi \left(\frac{N_b}{L_c} \right) \sum_{n=-\infty}^{\infty} F_1 \left(2\pi \frac{N_b}{L_c} n \right) \delta \left(k - 2\pi \frac{N_b}{L_c} n \right). \tag{22}$$

Inserting Eq.(22) into Eq.(8) we get:

$$\Phi(r,\phi,s) = \sum_{m=-\infty}^{\infty} e^{im\phi} \left(\int_0^b r_o dr_o \, \rho_{t,m}(r_o) \right) \frac{N_b}{L_c} \cdot \tag{23}$$

$$\sum_{n=-\infty}^{\infty} F_1 \left(2\pi \frac{N_b}{L_c} n \right) \tilde{G}_m \left(2\pi \frac{N_b}{L_c} n, r, r_0 \right) exp \left(i \, 2\pi \frac{N_b}{L_c} ns \right).$$

Note in passing that the $n = 0$ term in Eq.(23) gives *no* contribution to the wake-field, being (longitudinally) constant, and can be accordingly discarded. The sums in Eq.s(20), (22) and (23) can be truncated at $|n| \sim N_T$, where:

$$N_T \sim \frac{L_c}{2\pi N_b} \frac{\alpha}{\sigma_s}, \tag{24}$$

i.e. at the border of the (single) bunch spectrum $k_b \sim \alpha/\sigma_s$, where σ_s is the bunch length, and α is a factor of order one. The spectral argument k in $\tilde{G}_m(\cdot)$ and $F_1(\cdot)$ in Eq.(23) takes therefore only values that are integer multiples of the fundamental wavenumber:

$$k = n \left(\frac{2\pi N_b}{L_c} \right), \quad n = -N_T, \ldots, N_T, \tag{25}$$

Using the typical numbers, we get $0.\bar{6}\ m^{-1} \le k \le 13.\bar{3}\ m^{-1}$ for LHC, whereas for short-bunch small-ring machines, like $DAFNE$, $7.7\, m^{-1} \le k \le 50\, m^{-1}$.

CONCLUSIONS

In this paper we presented a rigorous computation of the Green's function for an (off-axis) multi-bunch beam running at constant velocity parallel to the axis of circular pipe with finite wall conductivity and thickness. More or less trivial extensions include more complicated geometries (e.g., elliptical, square). The solution is exact but not handy. Appropriate asymptotic forms are discussed in a companion paper.

ACKNOWLEDGEMENT

This work was made at CERN, in the SL/AP group. One of the authors (L.C.) wish to thank F. Ruggiero (CERN SL/AP) for several suggestions and advices.

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