EQUILIBRIUM LONGITUDINAL DISTRIBUTION FOR LOCALIZED REGULARIZED INDUCTIVE WAKE

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Abstract

We use the Gaussian approximation to confirm that, as noted for Haissinski's equation, a steady state solution for the longitudinal phase space distribution function always exists if a physically regularized inductive wake is used.

INTRODUCTION

In a recent paper [1] we have shown that assuming a localized wake and the Gaussian approximation for the longitudinal beam distribution function one can understand the nature of the stationary solutions for the inductive wake, by comparison between the resulting map and the Haissinski equation, which rules the (less realistic) case of a uniformly distributed wake. In particular we showed that the non-existence of solutions of Haissinski's equation when the inductive wake strength exceeds a certain threshold [2] corresponds to the onset of chaos in the map evolving the moments of the beam distribution from turn to turn. In this paper we use the same map to confirm that, as noted in [2] for Haissinski's equation, a steady state solution for the longitudinal phase space distribution function always exists if a physically regularized inductive wake is used.

THE MOMENT MAPPING

The longitudinal beam dynamics in electron storage rings can be described by the stochastic equations of motion for a single particle (Langevin equations). Introducing the canonical variables:

$$x_1 = \frac{longitudinal\ displacement}{natural\ bunch\ length},$$

$$x_2 = \frac{relative\ energy\ spread}{natural\ energy\ spread},$$

and integrating the Langevin equations over one turn, we obtain the following stochastic mapping:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = U \begin{pmatrix} x_1 \\ \Lambda x_2 + \hat{r}\sqrt{1-\Lambda^2} - \phi(x_1) \end{pmatrix},$$

where $\vec{X}' = (x_1', x_2')$ is $\vec{X} = (x_1, x_2)$ after one turn. Here U is the rotation matrix:

$$U = \begin{pmatrix} \cos\mu & \sin\mu \\ -\sin\mu & \cos\mu \end{pmatrix}, \tag{1}$$

$\mu = 2\pi\nu_s$, ν_s being the synchrotron tune, $\Lambda = \exp(-2/T)$, T being the synchrotron damping time measured in units of the revolution period, $\hat{r}$ is a Gaussian random variable with $<\hat{r}>= 0$ and $<\hat{r}^2>= 1$. The wake force $\phi(x_1)$ is represented by:

$$\phi(x_1) = \frac{Q_{tot}}{\sigma_0 E_0}\int_0^\infty \rho(x-u)W(u)du. \tag{2}$$

where E_0 is the nominal beam energy, σ_0 is the nominal relative energy spread ($\sigma_0 E_0$ is the natural energy spread), $W(x)$ is the wake potential and $\rho(x)$ is the charge density normalized to one. Note that synchrotron oscillations have been linearized, and radiation is localized at one point of the ring [3]. The above stochastic mapping is equivalent to an infinite hierarchy of deterministic mappings in the following statistical quantities: $\bar{x}_i =< x_i >$, $\sigma_{ij} =< (x_i - \bar{x}_j)(x_j - \bar{x}_j) >$, and so on, which are the moments of the distribution function $\psi(\vec{x})$, $< * >$ indicating an average over all particles. Our main assumption is that the distribution function in phase space is always a Gaussian, even in the presence of a wake force:

$$\psi(x_1, x_2) = \frac{\exp[\frac{1}{2}\sum_{i,j}^2 \sigma_{i,j}^{-1}(x_i - \bar{x}_i)(x_j - \bar{x}_j)]}{2\pi\sqrt{det\sigma}}. \tag{3}$$

We consider a purely inductive wake function

$$W(x) = b\delta'(x) \tag{4}$$

and split the mapping for the second order moments into three parts, representing the effect of radiation, wake-force and synchrotron oscillation, as follows:
radiation:

$$\begin{aligned} \sigma'_{11} &= \sigma_{11} \\ \sigma'_{12} &= \Lambda\sigma_{12} \\ \sigma'_{22} &= \Lambda^2\sigma_{22} + (1-\Lambda^2), \end{aligned} \tag{5}$$

wake force:

$$\begin{aligned} \sigma'_{11} &= \sigma_{11} \\ \sigma'_{12} &= \sigma_{12} + \frac{b}{4\sqrt{\pi\sigma_{11}}} \\ \sigma'_{22} &= \sigma_{22} + \frac{b\sigma_{12}}{2\sigma_{11}\sqrt{\pi\sigma_{11}}} + \frac{b^2}{6\sigma_{11}^2\pi\sqrt{3}}, \end{aligned} \tag{6}$$

synchrotron oscillation:

$$\sigma'_{ij} = \sum_{h,k=1}^{2} U_{ih}\sigma_{hk}U^t_{kj}. \tag{7}$$

The stability of the system depends on the values of the

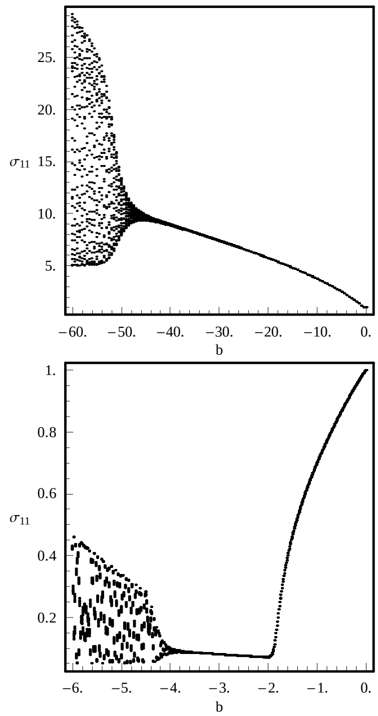

Figure 1: Purely inductive wake function: σ_{ij} versus b for $T = 200$, $\nu_s = 0.085$ with $N_S = 1$ (top) and $N_s = 150$ (bottom).

synchrotron tune ν_s, the damping time (measured in number of turns) T and the strength of the wake force b. We studied a wide range of parameters values and found stable solution of period-one and period-two, multi-stable states and coexistence of solutions with different periodicity. As an example we plot $\sigma_{ij}(b)$ in Fig. 1 (top) for $T = 200$ and $\nu_s = 0.085$. In the parameter space a chaotic region shows up and this behavior mainly depends on the b value but is almost indipendent on ν_s and T.

The localized wake approach can be extended to much more general cases, even uniformly distributed wakes, described by the Haissinski equation. To do so one should introduce the superperiodicity N_s and let it grow to infinity. This is done introducing in the mapping the following substitutions: $\nu_s \longrightarrow \nu_s/N_s, \quad T \longrightarrow TN_s, \quad b \longrightarrow b/N_s$ Computing $\sigma_{ij}(b)$ for different N_s and $b > 0$ we found that as N_s increases the mapping curves converge to the solution of the PWD equation.

In Fig. 1 (bottom) we show $\sigma_{ij}(b)$ with $N_s = 150$: the chaotic behavior exists also for $N_s >> 1$.

HAISSINSKI EQUATION

Introducing the syncrotron variables $\epsilon = E - E_s$ and $\tau = (z - z_s)/(\beta c)$, where E and z rispectively are the energy and longitudinal position of the single particle, the suffix s indicates the (synchronous) ideal particle with the nominal energy, β is the relativistic factor and c is the light velocity in vacuum. In the synchrotron phase space $\{\tau, \epsilon\}$ the single particle dynamics is described by the following equations

$$\dot{\tau} = -\frac{\alpha\epsilon}{E_s}, \dot{\epsilon} = \frac{eV_{RF}(\tau) - U_0 - D\epsilon}{T_0} - M(t) \qquad (8)$$

where α is the momentum compaction factor, $V_{RF}(x)$ the radio frequency accelerating potential, T_0 the particle revolution period, U_0 the energy lost at every turn, D the damping constant and $M(t)$ describes the quantum fluctuations. The total potential felt by the particle due to the accelerating structure and the wake field is

$$V(\tau) = V_{RF}(\tau) + \int_0^{\infty} eW(\tau)u(\tau - \xi)d\xi \qquad (9)$$

where $u(x)$ is the equilibrium longitudinal particle density

$$u(x) = \int_{-\infty}^{+\infty} \psi(x, y)dy \qquad (10)$$

which satisfies the normalization condition

$$\int_{-\infty}^{+\infty} u(x)dx = 1. \qquad (11)$$

Introducing the variables $x = \omega_0\tau/\sigma_0$, where $\sigma_0 = \sqrt{H_0 T_0/e\dot{V}_{RF}(0)}$, H_0 is the nominal particle energy, ω_0 the revolution frequency and letting $eV_{RF}(0) = U_0$ the Haissinski equation can be written as [4]:

$$u(x) = K \exp\left\{-\frac{x^2}{2} - \int_0^{+\infty} S(y)u(x - y)dy\right\}, \qquad (12)$$

where

$$S(y) = \int_0^{y\sqrt{2}\omega_0\sigma_0} w(\xi)d\xi, \qquad (13)$$

$$w(\xi) = \frac{e^2 N}{T_0 H_0} W(\xi), \qquad (14)$$

with total bunch particles N. Following [5] and letting $w(x) = B\delta'(x)$, we rewrite equation (12) as

$$\log u(x) + Bu(x) = \log K - \frac{x^2}{2}. \qquad (15)$$

The solution of this equaton is given in terms of the Lambert W-function [6], which is defined by the series expansion:

$$W_L(z) = \Sigma_{n=0}^{\infty} \frac{(-n)^{n-1}}{n!} z^n \qquad (16)$$

and is

$$u(x) = \frac{W_L[KB \exp(-x^2/2)]}{B} \qquad (17)$$

For $B \geq 0$ the solution of (12) exists always. On the other hands, for $B \leq 0$, in particular for

$$B \leq -\int_0^1 \frac{1 - x}{\sqrt{x - \log x - 1}} \sim -1.55061 \qquad (18)$$

no value of K can satisfy the normalization condition (11). If $B < -1.55061$ the sistem is unstable, therefore the solution of Haissinski equation does not exist. This result, well known in Literature [5], and recently studied in [2], is in very good agreement with the Gaussian approximation: there is a threshold value of b such that below this threshold (b_{thr}) the mapping gives a chaotic behavior for the second order moments. In Fig. 2 we show $b_{thr}(N_s)$: increasing the superperiodicity b_{thr} first shifts, then saturates approaching the threshold value $B = -1.55061$ given by the Haissinski equation.

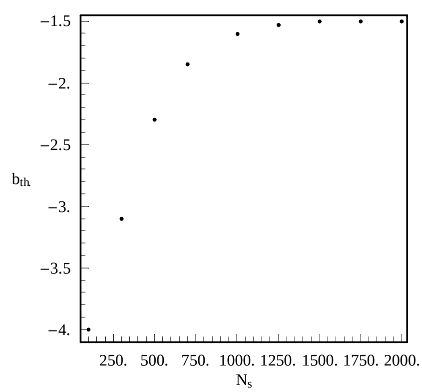

Figure 2: The threshold value of b versus N_s, for which the moments $\sigma_{ij}(b)$ become unstable and a chaotic regime shows up. We assume $T = 200$ and $\nu_s = 0.085$.

REGULARIZED INDUCTIVE WAKE

In a recent paper [2] one of the authors suggested to regularize the singularity of the wake function δ' in a physical way replacing the wake δ' with

$$W_r(x) = b \ \frac{\delta(x) - \delta(x-a)}{a}, \tag{19}$$

where a is a positive parameter, so as to satisfy the causality condition. Obviously when $a \to 0$ we recover the δ' function. The regularized wake (19) takes into account the fact that to compute the derivative of the δ function at x we need information around the neighborhood of x, but physically this is impossible, because each particle is influenced only by all those who preceede it but not follow. The mapping for the wake force is the following

$$\begin{aligned}
\sigma'_{11} &= \sigma_{11} \\
\sigma'_{12} &= \sigma_{12} + \frac{b\exp(-a^2/4\sigma_{11})}{4\sqrt{\pi\sigma_{11}}} \\
\sigma'_{22} &= \sigma_{22} + \frac{b\exp(-a^2/4\sigma_{11})\sigma_{12}}{2\sqrt{\pi\sigma_{11}}\sigma_{11}} \\
&\quad - \frac{b^2(1-\exp(-a^2/4\sigma_{11}))^2}{4a^2\sigma_{11}\pi} \\
&\quad + \frac{b^2(1+\exp(-a^2/\sigma_{11}) - 2\exp(-a^2/3\sigma_{11}))}{2\sqrt{3}a^2\sigma_{11}\pi},
\end{aligned} \tag{20}$$

We found that the chaotic behaviour disappears both for $N_s = 1$ and $N_s >> 1$ and the system is always stable, yelding periodic solutions in the parameters space. In Fig. 3 we show $\sigma_{11}(b)$ for $T = 200$, $\nu_s = 0.085$ with $N_s = 150$, using different values of a. Smaller values of the parameter a entail a behaviour closer to that one of the $\delta'(x)$ wake. The comparison of Fig. 1 and 3 shows that the regularization of the wake cancels the instabilities. Our results agree with those obtained in [2].

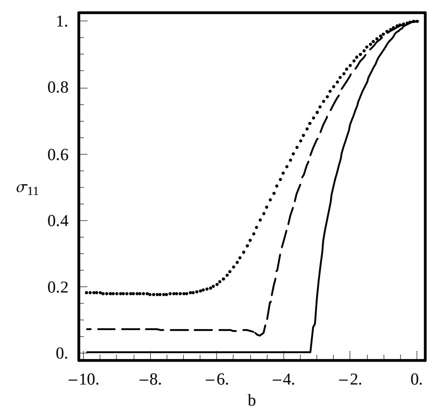

Figure 3: $\sigma_{11}(b)$ for $a = 0.1$ (dotted line), $a = 0.05$ (dashed line), $a = 0.01$ (solid line) using the regularized wake function with $T = 200$, $\nu_s = 0.085$ and $N_s = 150$.

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