TRANSFER MATRICES FOR THE COUPLED SPACE CHARGE DOMINATED SIX-DIMENSIONAL PARTICLE MOTION

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Abstract

In this paper we present exact analytical solutions for the particle motion in the six-dimensional phase space taking into account the space charge forces of linear coupled beam. The transfer matrices for the typical elements of magnetic lattice, such as drifts, cavities, quadrupole and dipole magnets have been obtained. The symplectic transfer matrices are used to develop a tracking program for the coupled betatron and synchro-betatron motion that enables the simulation of the tilted beam effects in circular accelerators.

INTRODUCTION

Intense, high-brightness electron beams foresee in third generation light sources require minimal emittance and lowest particle loss throughout their acceleration and transport. The dynamics of such beam is significantly affected by strong self forces due to space charge. We are interested in study of direct space charge effects on electron bunch motion in storage rings, such as detuning and excitation of additional resonance lines, change of betatronic oscillation amplitudes for the tilted in all three coordinate planes electron bunch. In conjunction with magnet non-linearities, all these effects may give rise to particle diffusion, to large betatron amplitudes and eventually to particle loss. For the tilted beam crossing a periodic lattice of storage ring the space charge induced coupling effects between the beam transverse coordinates and synchro-betatron coordinates are studied using element-by-element 6D matrix formalism. We present also some preliminary numerical results for the CANDLE [1] storage ring, based on computer code that uses obtained exact analytical solutions for the transfer matrices of common elements of magnetic lattice.

TRAJECTORY EQUATIONS

For particle motion description we use the Lagrangian formalism starting from the Lagrangian of a charged particle with the momentum p and charge e in electromagnetic field

$$L = -m_0 c^2 \sqrt{1 - \frac{1}{c^2}\left[\dot{x}^2 + \dot{y}^2 + \left(1 + h_x(s)\cdot x\right)^2 \dot{s}^2\right]} + e\left[\dot{x}A_x + \dot{y}A_y + \left(1 + h_x(s)\cdot x\right)\dot{s}A_s\right] - e\phi \qquad (1)$$

where (A_x, A_y, A_s) and ϕ are components of electromagnetic field vector potential and scalar potential, respectively, the derivations are taken with the respect to time variable.

From (1) one can derive the equation of particle trajectory in the natural coordinate system (x, y, z) connected with the design orbit in the form:

$$x' = p_x ,$$

$$p_x' = h_x(1+\delta) + \frac{e}{p_0}(1 + h_x \cdot x)\left[B_y - p_y B_s\right] + \frac{1}{cp_0} F_{x,s} ,$$

$$y' = p_y ,$$

$$p_y' = \frac{e}{p_0}\cdot\left(1 + h_x \cdot x\right)\cdot\left[p_x \cdot B_s - B_x\right] + \frac{1}{c\cdot p_0}\cdot F_{y,s} ,$$

$$z' = \frac{1}{\gamma^2}\cdot\delta - h_x(s)\cdot x \qquad (2)$$

$$\delta' = -\frac{e\cdot\omega_{RF}\cdot E_m(s)}{c^2\cdot p_0}\cdot\cos\left(\frac{\omega_{RF}}{c}\cdot\left(s - s_0\right) + \varphi\right)\cdot z + \frac{1}{c\cdot p_0}\cdot F_{z,s} ,$$

where $z = s - v_0 \cdot t$ is the longitudinal coordinate that defines the particle distance from the bunch centre, p_x, p_y are the normalized to design momentum transverse momenta, $\delta = \frac{\Delta E}{E_0}$ is relative energy deviation, $h_x(s) = -\frac{e}{p}\cdot B_y(0,0,s)$ is the trajectory curvature in the bending magnets, E_m, ω_{RF} are the accelerating field amplitude and frequency in the cavity, respectively, prime denotes derivative with the respect to longitudinal coordinate s. The last terms in the r.h.s of the second, fourth and sixth equations represent self-repulsive forces from the bunch space charge. We solve above equation in linear approximation for each element of storage ring magnetic lattice including only transverse components of magnetic field, neglecting the end field effects and applying linear optics technique (matrix formalism). To go forward we need also explicit expressions for bunch space charge forces. We dealt with this in next section.

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SPACE CHARGE FORCES

In calculation of space charge forces for simplicity we have made the following assumption about the particle distribution of the bunch:

a) elliptical cross sectional bunch with the uniform charge distribution within the bunch where space charge forces varies linearly with the transverse coordinates.
b) we consider the quasi-monochromatic beam with small momentum spread $\delta << 1$.

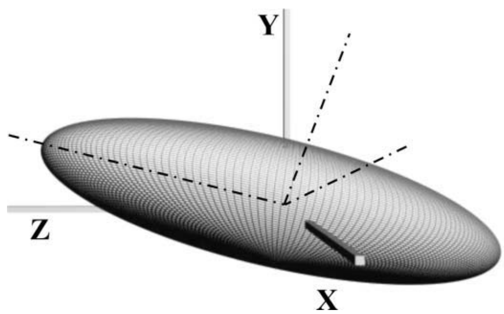

Figure 1. Schematic layout of tilted bunch.

Taking into account above assumptions, the space charge forces expressions [2] in the rest coordinate system of the tilted bunch can be transformed into the natural coordinate frame resulting to the following expressions:

$$F_{x,s} = F_{xx} \cdot x - F_{xy} \cdot y - F_{xz} \cdot z$$
$$F_{y,s} = -F_{yx} \cdot x + F_{yy} \cdot y + F_{yz} \cdot z \qquad (3)$$
$$F_{z,s} = -F_{zx} \cdot x + F_{zy} \cdot y + F_{zz} \cdot z$$

where

$$F_{xx} = \left(I_1 \cdot \cos^2\Theta_3 + I_3 \cdot \sin^2\Theta_3\right) \cdot \cos^2\Theta_1 + \frac{I_3 - I_1}{2} \cdot \sin 2\Theta_2 \cdot \sin\Theta_2 \cdot \sin 2\Theta_3 + \left[I_2 \cdot \cos^2\Theta_2 + \left(I_3 \cdot \cos^2\Theta_3 + I_1 \cdot \sin^2\Theta_3\right) \cdot \sin^2\Theta_2\right] \cdot \sin^2\Theta_1;$$

$$F_{xy} = -\left(I_1 \cdot \cos^2\Theta_3 + I_3 \cdot \sin^2\Theta_3\right) \cdot \sin 2\Theta_1 + \left(I_1 - I_3\right) \cdot \cos 2\Theta_1 \cdot \sin\Theta_2 \cdot \sin 2\Theta_3 - \left[I_2 \cdot \cos^2\Theta_2 + \left(I_3 \cdot \cos^2\Theta_3 + I_1 \cdot \sin^2\Theta_3\right) \cdot \sin^2\Theta_2\right] \sin 2\Theta_1; \qquad (4)$$

$$F_{xz} = \frac{\left(I_1 - I_3\right)}{2} \cdot \cos\Theta_1 \cdot \cos\Theta_2 \cdot \sin 2\Theta_3 + \left[I_2 - \left(I_3 \cdot \cos^2\Theta_3 + I_1 \cdot \sin^2\Theta_3\right)\right] \cdot \sin\Theta_1 \cdot \sin\Theta_2 \cdot \cos\Theta_2;$$

$$F_{yy} = \left[I_2 \cdot \cos^2\Theta_2 + \left(I_3 \cdot \cos^2\Theta_3 + I_1 \cdot \sin^2\Theta_3\right) \cdot \sin^2\Theta_2\right] * \cos^2\Theta_1 + \left(I_1 \cdot \cos^2\Theta_3 + I_3 \cdot \sin^2\Theta_3\right) \cdot \sin^2\Theta_1 + \frac{I_1 - I_3}{2} \cdot \sin 2\Theta_1 \cdot \sin\Theta_2 \cdot \sin 2\Theta_3;$$

$$F_{yz} = -\frac{\left(I_1 - I_3\right)}{2} \cdot \sin\Theta_1 \cdot \cos\Theta_2 \cdot \sin 2\Theta_3 + \left[I_2 - \left(I_3 \cdot \cos^2\Theta_3 + I_1 \cdot \sin^2\Theta_3\right)\right] \cdot \cos\Theta_1 \cdot \sin\Theta_2 \cdot \cos\Theta_2;$$

$$F_{zz} = \left(I_3 \cdot \cos^2\Theta_3 + I_1 \cdot \sin^2\Theta_3\right) \cdot \cos^2\Theta_2 + I_2 \cdot \sin^2\Theta_2;$$

$$F_{yx} = F_{xy}, F_{zx} = F_{xz}, F_{zy} = F_{yz};$$

where the space charge integrals are given by:

$$I_1 = \frac{e^2 \cdot N}{\pi \cdot \varepsilon_F \cdot l_b \cdot \gamma^2 \cdot a \cdot (a+b)};$$
$$I_2 = \frac{e^2 \cdot N}{\pi \cdot \varepsilon_F \cdot l_b \cdot \gamma^2 \cdot b \cdot (a+b)}; \qquad (5)$$
$$I_3 = \frac{e^2 \cdot N}{\pi \cdot \varepsilon_F \cdot \gamma^3 \cdot l_b^3} \cdot \ln \frac{\gamma \cdot l_b + \sqrt{\gamma^2 \cdot l_b^2 + a^2 - b^2}}{a+b};$$

with N the particle population per bunch, a and b the bunch ellipse half axes in horizontal and vertical planes (for normal oriented bunch) respectively, l_b the bunch length, γ the relativistic Lorenz factor and $\varepsilon_F = 8.85 \cdot 10^{-12} F/m$ the Faraday's constant. The $\Theta_1, \Theta_2, \Theta_3$ are angles between coordinate axis and the bunch symmetry axis projections onto the $(x-y);(y-z)$ and $(z-x)$ planes, respectively. They can be calculated via the ellipse characteristic parameters. For example, in the transverse plane one has [2]

$$\tan 2\Theta_1 = \frac{2E_x G_x}{E_x^2 - E_y^2} \qquad (6)$$

where the ellipse parameters are defined by the particle state vector components calculated for the given longitudinal position of the bunch. In order to approach self-consistent description of bunch motion within the any lattice element one can use slicing method, i.e. dividing the magnetic element to reasonable number of parts and then apply matrix manipulation technique.

ANALITICAL SOLUTIONS

We solve system of trajectory equations (2) in linear approximation after substitution of self-force expressions (3)-(4) and external magnetic field expressions in hard edge approximation. The obtained linear system of six first order differential equations we bring to the form of one differential equation of sixth order. Due to the fact that in this equation appears only the terms of even order, one can find the exact solution by searching it in the exponential form and then solving the corresponding algebraic equation. To save paper space we omit here the detailed mathematical manipulations and give the final result only for the first element of combined function bending magnet transfer matrix:

$$m_{11} = \frac{b_{11}(\xi_1 + b_{22}) - \xi_1\xi_2 + a_{12}a_{21} - h_x^2 a_{33}}{(\xi_1 - \xi_3)(\xi_1 + \xi_2)} ch\sqrt{\xi_1} l +$$

$$\frac{-b_{11}(\xi_2 - b_{22}) + \xi_1\xi_3 + a_{12}a_{21} - h_x^2 a_{33}}{(\xi_1 + \xi_2)(\xi_2 + \xi_3)} \cos\sqrt{\xi_2} l +$$

$$\frac{-b_{11}(\xi_3 + b_{22}) + \xi_1\xi_2 - a_{12}a_{21} + h_x^2 a_{33}}{(\xi_1 - \xi_3)(\xi_2 + \xi_3)} ch\sqrt{\xi_3} l; \qquad (8)$$

where $\xi_{1,2,3}$ are functions of parameters

$b_{11} = k_b - h_x^2 + a_{11}; b_{22} = k_b - a_{22};\ a_{11} = F_{xx,self} / E_0$

For verification of obtained analytical solutions we have checked it against single particle theory in the limit of zero particle population within the bunch. We have also re-calculated all optical parameters in this limit and found the consistency of analytical approach.

NUMERICAL RESULTS

Above approach was applied to beam dynamics study in the CANDLE storage ring, the main parameters of which are presented in the Table 1.

Table 1: CANDLE main parameters list.

Parameter	**Value**
Energy E (GeV)	3
Circumference (m)	216
Current I (mA)	350
RF frequency (MHz)	499.654
Harmonic number	360
Number of super-periods	16
Straight section length (m)	4.8
Lattice type	DBA
Bending radius ρ (m)	7.385

As show simulation results (see Fig. 2), more severe are limits to the particle population imposed by difference resonance of second order at the value of particle number per bunch $N_b = 2.72 \cdot 10^{11}$.

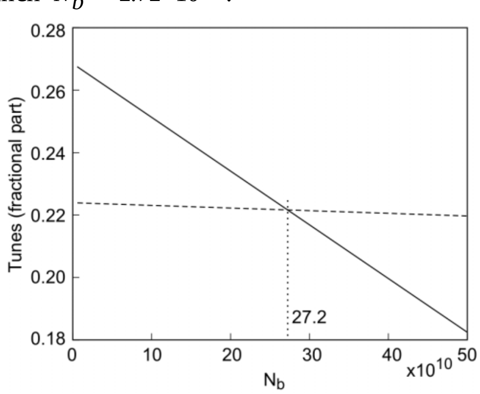

Fig.2. CANDLE tune dependence of particle population within the bunch.

As it was expected the space charge forces are stronger for vertical degree of freedom of particle motion. However, this limit on particle population per bunch is in two orders higher than the CANDLE design bunch population 0.56x10^{10}.

In the Fig.3 is shown the evolution of the transverse beam tilt angle (ignoring synchro-betatron coupling) in one lattice superperiod. As an initial tilt angle was taken 1mrad that approximately corresponds to the coupling value of one percent.

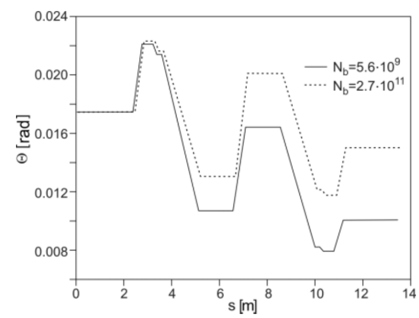

Fig .3. Bunch tilt angle variation within one superperiod.

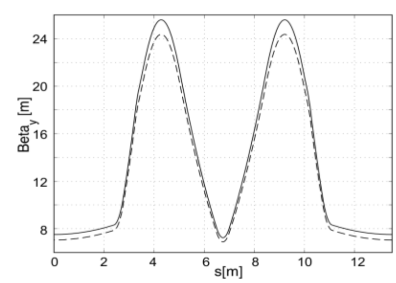

Fig.4. Beta beating of vertical oscillations due to space charge defocusing effect (dashed line - single particle motion, solid line – with the space charge).

Fig. 4 shows the beta-beating effect of space charge defocusing forces on the vertical oscillations.

CONCLUSION

Development of the computer code based on exact analytical solutions for the transfer matrices for the study of six dimensional fully coupled space charge dominated beam motion is under the progress. We hope after the completion of programme improvement to carry out detailed investigation of space charge dominated tilted beam induced various effects in circular accelerators and storage rings.

REFERENCES

[1] CANDLE- 3 GeV Synchrotron Light Source, Design Report, Yerevan, Armenia. 2002.
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