LONGITUDINAL SCHOTTKY SPECTRA OF BUNCHED BEAMS *

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Abstract

In this paper we derive an expression for the longitudinal Schottky power spectrum of a bunched beam in an arbitrary-shape stationary rf bucket. This, in turn, allows extract properties of the beam momentum distribution function. Several examples are also given in the paper.

INTRODUCTION

The beam current in a circular accelerator exhibits a random (Schottky) component due to its large but finite number of particles [1]. The Schottky noise signals have been a powerful diagnostics tool in many storage rings and synchrotrons. The theory of a Schottky power distribution in unbunched as well as bunched (by a linear rf voltage) beams is well understood [2]. In this paper we extend the theory to beams, bunched by an arbitrary wave-form stationary rf voltage. This was needed to analyze the antiproton and proton beams in the Fermilab's Recycler ring. The Recycler ring is a fixed-energy 8.9-GeV/c, 3.3-km antiproton storage ring. The Recycler rf system employs a broadband cavity to bunch the beam with an arbitrary-shape rf voltage [3]. Typically, the Recycler beam is bunched longitudinally by a barrier-bucket rf waveform. Under certain bucket conditions, the dependence of the synchrotron frequency on the particle energy becomes non-monotonic. It complicates the Schottky spectrum derivation and interpretation; we address these difficulties at the end of our paper.

ANALYSIS

The current of a bunched beam can be written in a form:

$$J(t) = e\omega_0 \sum_{n=1}^{N} \delta_{2\pi}\big(\omega_0 t + \theta(\Omega_n, \phi_n + \Omega_n t)\big) \qquad (1)$$

$$= ef_0 \sum_{n=1}^{N} \sum_{h=-\infty}^{\infty} \exp\big(ih\omega_0 t + ih\theta(\Omega_n, \phi_n + \Omega_n t)\big),$$

where $\omega_0 = 2\pi f_0$ is the frequency of the fundamental rf harmonic, N – the number of particles, θ – the azimuthal deviation of the n-th particle from the bunch center, which depends on synchrotron frequency, Ω_n, and the initial phase of synchrotron oscillations, ϕ_n. A well-known expansion of periodic δ-function in Fourier series is used, generating an infinite set of harmonics; however, only a portion of them can be actually resolved by the Schottky monitor. Then the spectrum is transformed by some analog or digital narrow-band filter with its central frequency, ω_f, generating a signal:

$$J_{\omega_f}(t) = \int_{-\infty}^{t} F_{\omega_f}(t-t')\, J(t')\, dt', \qquad (2)$$

where t is the time when the measurement is ended. The power of this signal is:

$$P_{\omega_f} = R\,(ef_0)^2 |J_{\omega_f}|^2 = R\,(ef_0)^2 \times \qquad (3)$$

$$\Big|\int_{-\infty}^{t} F_{\omega_f}(t-t')\, dt' \sum_{n,h} \exp\big(ih\omega_0 t' + ih\theta(\phi_n + \Omega_n t')\big)\Big|^2,$$

where R is the input resistance of the spectrum analyzer. The Schottky signal is a portion of the power associated with individual particles (no interference), which can be written in a form:

$$P^{(S)}_{\omega_f} = R\,(ef_0)^2 \int_{-\infty}^{t}\int_{-\infty}^{t} F_{\omega_f}(t-t') F^*_{\omega_f}(t-t'')\, dt' dt'' \times$$

$$\sum_{h',h''} \exp\big(i\omega_0(h't' - h''t'')\big) \times \qquad (4)$$

$$\sum_{n} \exp\big(ih'\theta(\phi_n + \Omega_n t') - ih''\theta(\phi_n + \Omega_n t'')\big).$$

Because the number of particles is very large, the sum over n can be replaced by an integral according to the formal scheme:

$$\sum_{n=1}^{N} = \int_0^N dN = \int_0^\infty N'(A)\, dA \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} \qquad (5)$$

$$= \int\int N'\big(A(\theta, p)\big)\, d\theta dp,$$

where A is the action, which is the area enveloped by a phase trajectory, and $N'(A)$ is the corresponding distribution function. It should be taken into account also that the bandwidth of the filter is always much smaller than the revolution frequency, which suppresses the contribution of harmonics with different h in Eq. (4), resulting in a general formula for the Schottky spectrum:

$$P^{(S)}_{\omega_f} = R\,(ef_0)^2 \sum_h \int\int N'(A)\, dA \frac{d\phi}{2\pi} \times \qquad (6)$$

$$\int_0^\infty G_{\omega_f}(t) \exp\Big(i\, h\omega_0 t + ih\theta(\phi + \frac{\Omega t}{2}) - ih\theta(\phi - \frac{\Omega t}{2})\Big)\, dt,$$

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where

$$G_{\omega_f}(t) = 2\ \mathrm{Re} \int\limits_0^\infty F^*(t')F(t+t')\,dt'. \qquad (7)$$

General formula (6) can be significantly simplified in some cases. The function $G(t)$, oscillating with frequency ω_f, is actually negligible at $t \gg T$, where T is some characteristic time of the filter: the filling time for an analog filter, or the duration of measurements for a digital one. With the condition $\Omega T \ll 1$, Eq. (6) becomes particularly simple:

$$P^{(S)}_{\omega_f} \simeq RN\,(ef_0)^2 \int\limits_{-\infty}^{\infty} K(\omega_f, p)\,W(p)\,dp, \qquad (8)$$

where $f(p) = \omega(p)/2\pi$ is the revolution frequency of a particle with momentum p, and

$$W(p) = \frac{1}{N}\int N'\big(A(\theta,p)\big)\,d\theta \qquad (9)$$

$$K(\omega_f, p) = \sum_h \int\limits_0^\infty G_{\omega_f}(t)\exp\big(i\,h\omega(p)t\big)\,dt. \qquad (10)$$

It is easy to see that $W(p)$ is the normalized distribution function of the bunch in momentum, and the kernel $K(\omega_f, p)$ transforms it to some distribution in frequency. The following examples demonstrate that the kernel has a sharp maximum, and when used in Eq. (10), separates a part of the spectrum, where the revolution frequency multiplied by h, coincides with the frequency of the filter. Note that with $\omega_0 T \gg 1$ there in no overlapping of harmonics with different h. Such a regime of measurement is especially convenient to recreate the instant distribution in momentum.

As a first example, consider the usual LCR filter with the transfer function:

$$F_{\omega_f}(t) = \exp\left(-\Delta\omega_f t\right)\ \sin\left(\omega_f t\right) \qquad (11)$$

(in this case $1/\Delta\omega_f$ can be identified with the previously used T). With $\Delta\omega_f \ll \omega_f$ it gives the following function G and kernel K:

$$G_{\omega_f}(t) = \frac{1}{4}\,\exp\left(-\Delta\omega_f t\right)\ \cos\left(\omega_f t\right), \qquad (12)$$

$$K(\omega_f, p) = \frac{1}{4}\,\frac{\Delta\omega_f}{\big(h\omega(p) - \omega_f\big)^2 + \Delta\omega_f^2}. \qquad (13)$$

The latter expression can be written in a form:

$$K(\omega_f, p) \propto \frac{1}{\pi}\,\frac{\Delta p_f}{(p-p_f)^2 + \Delta p_f^2} \qquad (14)$$

where $p_f = (\omega_f - h\omega_0)/h\omega'_p$, $\Delta p_f = \Delta\omega_f/h\omega'_p$. This function tends to $\delta(p - p_f)$ at $\Delta p_f \to 0$ resulting in an exact recreation of the instanteneous momentum distribution by Eq. (8): $P^{(S)}_{\omega_f} \propto W(p_f)$. Fig.1 demonstrates the

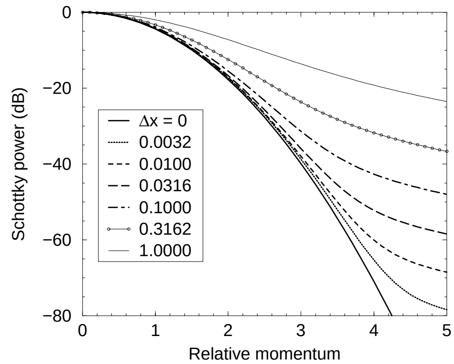

Figure 1: The Schottky spectrum with an LCR filter vs p_f/σ_p. at different $\Delta x = \Delta\omega_f/h|\omega'_p|\sigma_p$.

distortion of the original Gaussian spectrum with dispersion σ_p because of the nonzero filter bandwidth. The ratio p_f/σ_p is used for the horizontal axis and the relative Schottky signal as the vertical for various $\Delta p_f/\sigma_p$.

A digital filter with the uniform window is considered as another example. Its transfer function and kernel are:

$$F_{\omega_f}(t) = \frac{T}{M}\sum_{m=0}^{M-1}\delta\big(t - \frac{mT}{M}\big)\exp\big(2\pi i k\frac{m}{M}\big) \qquad (15)$$

$$K(\omega_f, p) \simeq \frac{T\sin^2\big(h\omega(p) - \omega_{fk}\big)\frac{T}{2}}{M^2\sin^2\big(h\omega(p) - \omega_{fk}\big)\frac{T}{2M}} \qquad (16)$$

where M is the number of points measured, and $\omega_{fk} = 2\pi k/T$. At $T \to \infty$, this expression also tends to δ-function like (14). The spectrum distortion with $M = 1024$ and the finite T is shown in Fig.2, where the same notations as in Fig.1 are used.

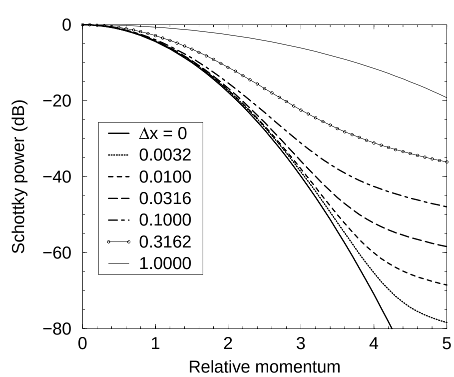

Figure 2: The Schottky spectrum with a digital filter vs p_f/σ_p for various $\Delta x = 2/h|\omega'_p|T\sigma_p$.

So, as one could expect, this regime of measurements allows to recreate the momentum spectrum with the accuracy being higher for a lower filter bandwidth (longer time of measurement). However, a very small bandwidth leads to a violation of the condition $\Omega T \ll 1$, resulting in a complicated split of the spectrum in harmonics of the synchrotron frequency. Formula (8) is inapplicable in such a case; nevertheless an exact recreation of the momentum spread is possible still. To prove it, it is better to rewrite Eq. (6) in a form:

$$P^{(S)}_{\omega_f} = R\,(ef_0)^2 \sum_h \int N'(A)\,dA \sum_{m=-\infty}^{\infty} |I_{mh}(A)|^2 \times \int_0^{\infty} G_{\omega_f}(t) \exp\big(i\,t\,(h\omega_0 + m\Omega)\big)\,dt, \qquad (17)$$

where

$$I_{mh}(A) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\big(ih\theta(A,\phi) - im\phi\big)\,d\phi. \qquad (18)$$

We will use the transfer function (11-12) at $\Delta\omega_f = 0$ corresponding to an ideal narrow-band filter. Then Eq. (17) gives:

$$P^{(S)}_{\omega_f} = \frac{\pi R}{4}\,(ef_0)^2 \sum_h \int_0^{\infty} N'(A)\,dA \sum_{m=-\infty}^{\infty} |I_{mh}(A)|^2 \times \delta\big(\omega_f - h\omega_0 - m\Omega(A)\big) \qquad (19)$$

At any h the spectrum contains a lot of lines for different m. The availability of the δ-function allows to integrate over A; however it is necessary to take into account a possible non-monotonic dependence of $\Omega(A)$. Such a situation is peculiar to the Recycler, where the accelerating voltage is a succession of rectangular pulses ('barriers') of alternating polarity. The synchrotron frequency, Ω, is a growing function at small A, however for some A it decreases because of deep ingress of particles into the barriers. Therefore, two values of $A_{1,2}$ may correspond to a given Ω, and Eq. (19) gives:

$$P^{(S)}_{\omega_f} = \frac{\pi R}{4}\,(ef_0)^2 \sum_h \sum_{m=-\infty}^{\infty} \sum_{i=1}^{2} \qquad (20)$$
$$|I_{mh}\big(A_i(\Omega_m)\big)|^2 N'\big(A_i(\Omega_m)\big) A_i'(\Omega_m),$$

where $\Omega_m = (\omega_f - h\omega_0)/m$. Further calculation is rather cumbersome, using specific parameters of the accelerator, etc. However, it is possible to get a very simple general expression for the measured dispersion of the momentum distribution, obtained at each h. Preliminary note that

$$\sum_{m=-\infty}^{\infty} |I_{mh}(A)|^2 = 1 \qquad (21)$$

and

$$\sum_{m=-\infty}^{\infty} |mI_{mh}(A)|^2 = \frac{h^2}{2\pi} \int_{-\pi}^{\pi} \left[\frac{\partial\theta}{\partial\phi}(A,\phi)\right]^2 d\phi \qquad (22)$$

Using this, one can show that at some h, the function (19) describes a distribution with the central frequency $h\omega_0$ and the dispersion:

$$\sigma_\omega^2 = \frac{h^2}{2\pi N} \int_{-\pi}^{\pi} \int_0^{\infty} \left[\frac{\partial\theta}{\partial\phi}(A,\phi)\Omega(A)\right]^2 N'(A)\,dA d\phi \quad (23)$$

Taking into account that

$$\Omega\,\frac{\partial\theta}{\partial\phi} = \frac{d\theta}{dt} = \omega(p) - \omega_0 = (p - p_0)\,\frac{d\omega}{dp}, \qquad (24)$$

as well as $dAd\phi/2\pi = d\theta dp$ and definition (9), one can rewrite this as

$$\sigma_\omega^2 = h^2 \left(\frac{d\omega}{dp}\right)^2 \int_{-\infty}^{\infty} W(p)\,(p - p_0)^2\,dp, \qquad (25)$$

which just gives the exact value for the momentum dispersion.

CONCLUSIONS

The general formula (6) for the Schottky spectrum of arbitrary bunched beams was provided. The most important conclusions from it are:

1. If the Schottky noise signal is registered with a band-pass filter wide enough to include several synchrotron frequencies and the time of measurement is short enough as compared to the synchrotron period, the Schottky spectrum represents an instantaneous "snap-shot" of the momentum distribution in a bunched beam. Deviations of the measured Schottky signal spectrum from an actual momentum distribution function depend on the characteristics of the band-pass filter as demonstrated in Fig. 1 and 2.

2. If the band-pass filter becomes very narrow, Eq. (20) gives a prescription on how to derive the Schottky power spectrum in any given case. A possible non-monotonic dependence of the synchrotron frequency on amplitude is taken into account also.

3. The general expression for the rms momentum measured by ultimately narrow-band filter was obtained and given by Eq. (25). It is interesting and important to point out that the r.m.s width of the Schottky spectrum in this case is always proportional to the r.m.s momentum spread of the beam.

ACKNOWLEDGEMENTS

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