# ANALYTICAL MODELS FOR CYCLOTRON ORBITS

H.L. Hagedoorn

Eindhoven University of Technology, the Netherlands

Abstract

A short review of analytical methods is presented for the treatment of orbit dynamics of particles not interacting with each other or with walls. The hamiltonian theory is used. For the acceleration a smooth approximation is used as well as an approximation in which the RF-structure is taken into account. A few examples of analytical calculations are given.

## 1) Introduction

In the proceedings of the international conferences on cyclotrons much has been published on the theoretical description of the orbit behaviour in cyclotrons. In the last conference a review article giving insight in many phenomena has been written by Lapostolle [1]. Initially much attention has been paid in the past to time independent orbit behaviour [2,3,4]. Acceleration effects were treated separately [5,6] or by slowly changing the relevant parameters in the time independent theory [7]. An example of the influence of the structure of the accelerating system on the orbits is the gap crossing resonance first mentioned by Gordon [5]. During the last years developments of a general theory including acceleration have made progress [8,9,10].

In this contribution a review of analytical methods is presented for the treatment of orbit dynamics of particles not interacting with each other or with walls. The classical hamiltonian theory is used. The resulting analytical models do not have the purpose to replace numerical calculations. They may be used, however, as a means to check numerical calculations in idealized cases and as a guide for understanding special effects as well as for finding corrections in the case of unwanted numerically or experimentally observed perturbations [11,12]. Numerical calculations have always to be carried out when high accuracy is needed or when the electric and magnetic fields are strongly nonlinear: i.e. for isochronism, the shape of extracted orbits, the central region, the extraction region.

As input for the analytical model equations generally only a few components of the fourier expansions of the electric and magnetic fields are needed. Often it turns out to be convenient to use also numerically found quantities like the radial and axial oscillation frequencies. All parameters are taken as functions of the radius. The model equations are sometimes slightly too complicated to obtain immediate results. Then simple computer programs may be needed. This occurs for example when varying parameters must be taken into account during the passage through a resonance or when two resonances lie closely to each other. Simple theoretical expressions often can only be found for slowly changing parameters when resonances are treated separately.

It is important to choose coordinates which give an easy access to the problem or which give a rough understanding of the orbit behaviour. Such coordinates may be cylindrical coordinates for the treatment of the radial motion. These coordinates are quite often finally abandoned after a number of transformations at the favour of Cartesian coordinates to show the motion of the orbit centre. When coupling between two oscillation modes is considered it is convenient to use action and angle variables which may act for our feelings as energy and phase of the oscillations.

For low energy particles a relativistic treatment is not strictly necessary. For particles in ring accelerators with cavities one has generally to start from a complete relativistic hamiltonian.

In this contribution the required successive transformations needed for simplifying the problem will be presented together with some first results. Then smooth acceleration will be considered followed by a treatment in which the realistic accelerating structure is taken into account. The theory will be applied to build up hamiltonians describing several perturbing effects together. Finally the start of the relativistic hamiltonian treatment for orbits on large radii will be demonstrated.

## 2) A first approximation of the radial motion

In a cylindrically symmetric magnetic field $B(r)$ a possible trajectory for charged particles with linear momentum $p_o$ and charge Ze is a circle with radius $r_o$: $r_o = p_o/ZeB(r_o)$. Particles with the same momentum but not moving on this circle will oscillate around it. If they move outside the circle they will experience a deviating magnetic induction and will therefore follow an orbit with a radius of curvature that differs from $r_o$. As a result they cross the circle at intervals which differ from $\pi$ radians. In this way a visual picture of the orbit trajectories is made and radial oscillations are shown. A particle trajectory has a radial distance x and a difference in direction $\alpha$ with respect to the circle at a given azimuthal position. Moving over an azimuthal interval $d\theta$ then two differential equations for x and $\alpha$ follow:

$$dx = (r_o + x)\,\alpha d\theta, \quad d\alpha = -(\omega dt - \omega_o dt_o) \qquad (1)$$

In these equations $\omega$ is the local revolution frequency of the deviating particle at azimuth $\theta$ and dt is the time needed for moving over $d\theta$. The index o is used for the particle on the circle. With the velocity $V_o = p_o/m$ and the field index $n = -(r/B \cdot dB/dR)_o$ one finds:

$$dt = \frac{r_o + x}{v_o}\, d\theta, \quad \omega = \frac{eB(r_o)}{m}\left(1 - \frac{x}{r_o} n\right)$$

Substitution in the orbit equations (1) yields the well known equation for radial oscillations in a cylindrically symmetric field:

$$\frac{d^2x}{d\theta^2} + (1-n)\, x = o, \quad \upsilon_R^2 = 1-n$$

This result follows also from an equation in which the radial acceleration is calculated from the unbalance between the centrifugal force and the Lorentz force when the radial position r is put equal to $r = r_o + x$ and the resulting equation is linearized [13].

The given equations are simple and follow from direct visual observations.
The reference orbit is circular and a linear approximation is used. In reality the reference orbit may have a complicated structure, nonlinear effects may play an important role in the excitation of resonances, acceleration in electric and time dependent magnetic field take place, coupling between radial, axial and longitudinal oscillations occurs, etc. Sometimes nevertheless the visual picture can be developped further.
However, it will become difficult to get reliable quantitative results. Moreover the use of models from a visual picture may lead to a description in which a number of essential effects are neglected. The reverse way in which one starts with a general theoretical model and then constructs a visual picture is much better. All these arguments point to general methods to investigate the orbit behaviour.

It is quite normal to connect the quantities x and $\alpha$ in cyclotrons with orbit centre coordinates. If a Cartesian coordinate system is oriented such that the X-axis coincides with the azimuthal direction $\theta$, then the orbit centre coordinates are given by

$$X = x\ (\theta), \quad Y = r_o \alpha\ (\theta) \qquad (2)$$

The direction deviation $\alpha$ equals in first approximation the quotient of the radial momentum $p_r$ and the total momentum $p_o$.

The notion of the orbit centre is still kept for non cylindrically symmetric fields. If the field has an N-fold symmetry there exists a closed orbit with mean radius $r_o$ having the same N-fold symmetry as the field but with a momentum that slightly differs from $P = e\langle B(r_o,\theta)\rangle r_o$ [4]. This difference is due to a small change in the circumference of the orbit with respect to the circle with radius $r_o$. The symmetry point of this N-fold symmetric closed curve is taken as orbit centre and the relations (2) are still sufficient for our understanding, though in the rigorous theory they are modified!

3) The hamiltonian representation in time independent fields

In time independent fields the hamiltonian representing the particle motion is given by:

$$H = \frac{1}{2m}\,(\underline{p} - e\underline{A})^2 + eV \qquad (3)$$

The magnetic vector potential and the electric scalar potential are given by $\underline{A}$ and V resp. As there is no explicit time dependency the energy E = H = constant. From this hamiltonian the equations of motion follow with time as the independent variable. In these cases it is often convenient to use one of the coordinates as independent variable and its conjugate momentum as the negative of a new hamiltonian [14-16]. The resulting equations of motion then describe the geometrical orbit shape.
In the case of a circular accelerator the azimuth $\theta$ will generally act as the new independent variable and $-p_\theta$ as the new hamiltonian.

Note that $p_\theta$ is found by algebraically solving it from the original hamiltonian.
In the case of ion optical devices with straight optical axes the coordinate z along these axes may be taken as the new independent variable and $-p_z$ as the new hamiltonian. Further it is advisable to use relative coordinates and momenta in many cases: e.g. $\xi = x/r_o$ and $\pi_r = p_r/p_o$. The hamiltonian has then to be scaled also. For the just mentioned example it follows that $\bar{H} = H/p_o r_o$. Application of the two procedures often leads to results.

*The solenoidal lens.* In a solenoidal lens with its axis along the z axis the vector potential is given by

$$A_\theta = -\frac{1}{2} rB_o + \frac{1}{16} r^3 \left(\frac{d^2B}{dz^2}\right)_o, \quad A_r = o, \quad A_z = o.$$

$B_o(z)$ and $(d^2B/dz^2)_o$ are the magnetic induction and its second derivative along the z axis. The hamiltonian is given by:

$$H = \frac{1}{2m}\left(\frac{p_\theta}{r} - eA_\theta\right)^2 + p_z^2 + p_r^2, \quad p_o = \sqrt{2mH}.$$

Taking $-p_z$ as the new hamiltonian, $\pi_\theta = p_\theta/p_o$ and $\pi_r = p_r/p_o$ as relative momenta, omitting constant terms, expanding the square root and keeping only second degree terms one finds the new hamiltonian:

$$K = -\pi_z = \frac{1}{2}\left(\frac{\pi_\theta}{r} - \frac{eA_\theta}{p_o}\right)^2 + \frac{1}{2}\pi_r^2$$

As there is no explicit $\theta$-dependency the momentum $\pi_\theta$ is a constant of motion. For a trajectory crossing somewhere the axis $\pi_\theta = o$. For such a trajectory the hamiltonian for paraxial rays becomes

$$K = \frac{1}{2}\frac{e^2A_\theta^2}{p_o^2} + \frac{1}{2}\pi_r^2 = \frac{1}{2}\left(\frac{1}{4}\frac{e^2B_o^2}{p_o^2}\right) r^2 + \frac{1}{2}\pi_r^2,$$

where the vector potential $A_\theta$ is substituted. The lens strength for thin lenses then equals $1/F = e^2B^2L/4p_o^2$ with L the length of the lens (see for more details e.g. [17]).

*The Wien filter.* A Wien filter has been used at Saclay[18] and Eindhoven for radial injection of polarized ions. For the treatment a Cartesian coordinate system will be used in which the optical axis is chosen along the Z-axis. The electric field points in the X-direction. The magnetic field is e.g. the cyclotron field. The series expansion of the electric potential is given by:

$$V(x,y) = x\left.\frac{\partial V}{\partial x}\right|_{x=y=o} + \frac{1}{2}x^2\left(\frac{\partial^2 V}{\partial x^2}\right)_{x=y=o} - \frac{1}{2}y^2\left(\frac{\partial^2 V}{\partial x^2}\right)_{x=y=o}$$

The magnetic vector potential is given by:

$$A_z = -B_y x - \frac{1}{2}x^2\left(\frac{\partial B_y}{\partial x}\right)_{x=y=o} + \frac{1}{2}y^2\left(\frac{\partial B_y}{\partial x}\right)_{x=y=o}, A_x = o, A_y = o$$

The hamiltonian equals

$$H = \frac{1}{2m}\left(p_x^2 - p_y^2 + (p_z - eA_z)^2\right) + eV, \qquad p_o = \sqrt{2mH},\ H = E$$


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Using relative momenta $\pi_x = p_x/p_o$, $\pi_y = p_y/p_o$ the new hamiltonian $-p_z$ up to second degree in the variables after expansion of the square root becomes:

$$K = x\,\frac{1}{2E}\left(\frac{\partial V}{\partial x}\right)_o - \frac{eB_y}{p_o}\,x$$
$$+\frac{1}{2}\pi_x^2 + \frac{1}{2}x^2\left(\frac{1}{4E^2}\left(\frac{\partial V}{\partial x}\right)_o^2 + \frac{1}{2E}\left(\frac{\partial^2 V}{\partial x^2}\right)_o + \frac{eB_y}{p_o}\left(\frac{1}{B_y}\frac{\partial B_y}{\partial x}\right)_o\right)$$
$$+\frac{1}{2}\pi_y^2 + \frac{1}{2}y^2\left(-\frac{1}{2E}\left(\frac{\partial^2 V}{\partial x^2}\right)_o - \frac{eB_y}{p_o}\left(\frac{1}{B_y}\frac{\partial B_y}{\partial x}\right)_o\right)$$

We now choose the electric field strength on the axis such that $(\partial V/\partial x)_o = -vB_y$ $(x=y=o)$. Due to this choice the first degree part in the hamiltonian disappears. The remaining second degree part represents oscillations around the optical axis with frequencies

$$\nu_x^2 = \frac{1}{4E^2}\left(\frac{\partial V}{\partial x}\right)_o^2 + \frac{1}{2E}\left(\frac{\partial^2 V}{\partial x^2}\right)_o + \frac{eB_y(0)}{p_o}\left(\frac{1}{B_y}\frac{\partial B_y}{\partial x}\right)_o$$
$$\nu_y^2 = -\frac{1}{2E}\left(\frac{\partial^2 V}{\partial x^2}\right)_o - \frac{eB_y(0)}{p_o}\left(\frac{1}{B_y}\frac{\partial B_y}{\partial x}\right)_o$$

For stability $\nu_x^2$ and $\nu_y^2$ have to be positive. Given the magnetic field of the cyclotron the magnetic contributions to the focussing are relatively small. For stability in both directions the electric field strength must have a linearly increasing value in the X-direction which may not be too large. This is rather similar to the condition for the value of the field index in classical cyclotrons.

## 4) The radial motion in a cyclotron without acceleration

Following the lines of the preceding section the negative of the azimuthal canonical momentum is taken as a new hamiltonian with the azimuth $\theta$ as independent variable. Also relative coordinates and momenta are used. The magnetic vector potential up to second degree in the axial coordinate z is described by:

$$A_\theta = \frac{1}{2}z^2\left(\frac{\partial B}{\partial r}\right)_{z=0} - \frac{1}{r}\int^r rB_o\,dr + \ldots,\quad A_r = \frac{1}{2}\frac{z^2}{r}\left(\frac{\partial B}{\partial \theta}\right)_{z=0} + \ldots,\quad A = o.$$

The hamiltonian then becomes (see ref. 4, e.g. 2.3):

$$K = -(1+x)\left(1-\left(\pi_r + \frac{1}{2}\frac{\zeta^2}{1+x}\frac{r_o}{p_o/e}\left(\frac{\partial B}{\partial \theta}\right)_o\right)^2 - \pi_z^2\right)^{\frac{1}{2}} + \quad (4)$$
$$\frac{r_o}{p_o/e}\int(1+x)B_o\,dx - \frac{1}{2}\zeta^2(1+x)\frac{r_o}{p_o/e}\left(\frac{\partial B}{\partial x}\right)_o,\quad x=\frac{r-r_o}{r_o},\ \zeta=\frac{z}{r_o}$$

The magnetic field in the median plane is represented by

$$B = B(r_o)\left(1+\mu' x + \frac{\mu'' x^2}{2} + \Sigma\left(A_k + xA'_k + \frac{x^2 A''_k}{2} + \ldots\right)\cos k\theta + \Sigma\left(B_k + B'_k + \frac{x^2 B''_k}{2} + \ldots\right)\sin k\theta\right) \quad (5)$$

The magnetic field has an N-fold symmetry. The summations should be carried out with $k = nN$.

As also imperfections are taken into account the summation is taken for all integer values of k, keeping in mind that only those which are a multiple of the field symmetry N are large. In a cylindrically symmetric field the second degree part in the hamiltonian immediately gives the betatron frequencies in classical cyclotrons. For the study of the radial motion without coupling the terms in $\zeta$ are skipped. Substitution of the representation for the magnetic field (5) then gives the radial hamiltonian $K_R$. Expanding the square root yields

$$K_R = x\left(\Sigma(A_n\cos n\theta + B_n\sin n\theta)\right) + \frac{1}{2}x^2\left(1+\mu' + \Sigma(A_n + A'_n)\cos n\theta + (B_n + B'_n)\sin n\theta\right) + \frac{1}{2}\pi_r^2 + \quad (6)$$
$$\frac{1}{2}\pi_r^2 x + \frac{1}{6}x^3\,(2\mu' + \mu'' + \text{Fourier terms}) + \ldots$$

$K_R = K_1 + K_2 + K_3 + \ldots$, in which $K_i$ represents the terms of degree i in the hamiltonian $K_R$. The aim now is to substract a reference orbit from the motion and to remove by a series of successive canonical transformations the $\theta$-dependency. As reference orbit is taken the orbit that has the same N-fold symmetry as the unperturbed magnetic field. The final hamiltonian then has the shape $K_R = K_R(X,Y) = $ constant, with X and Y the orbit centre coordinates expressed in e.g. mm. The function $K_R$ describes the flow lines in phase space. Quite generally the hamiltonian can be written as

$$K(X,Y) = \Sigma\alpha_{lk}\,X^l Y^k \quad (7)$$

The coefficients $\alpha_{lk}$ are composed out of constants and field quantities like the Fourier coefficients. The radial oscillation frequency can be either expressed in these field quantities or taken from a table. The flow lines follow from a very simple computer program using the equations of motion:

$$\dot{X} = \frac{\partial K}{\partial Y},\quad \dot{Y} = -\frac{\partial K}{\partial X} \quad (8)$$

The results show the slowly moving orbit centre instead of the complete complicated orbit shape. We will now shortly follow the arguments for the various transformations. The first degree part in the hamiltonian is removed mainly by a suitable choice of a generating function which yields a simple coordinate transformation

$$\bar{x} = x - x_{eq},\quad \bar{\pi}_r = \pi_r - \pi_{eq} \quad (9)$$

The functions $x_{eq}$ and $\pi_{eq}$ are $\theta$-dependent and represent the equilibrium orbit, i.e. the closed orbit in the unperturbed N-fold symmetric field. Terms in $K_1$, which have a Fourier frequency in the neighbourhood of the expected radial frequency should never be removed. For easiness we will omit below the bars above new quantities.

The second degree part of the hamiltonian will now be brought to the so-called normal shape [19,4]:

$$K_2 = K_{20}x^2 + K_{11}\,x\,\pi_r + K_{02}\,\pi_r^2 \rightarrow K_2 = \frac{1}{2}\pi_\theta^2 + \frac{1}{2}Q(\theta)\,x^2$$

This transformation is not strictly needed but it may simplify the administration. The introduction of action and angle variables $(I,\phi)$ gives access to transformations which can remove the azimuthal dependency:

$$\pi_r = (2I\upsilon_r)^{\frac{1}{2}} \sin\phi,\ x = \left(\frac{2I}{\upsilon_r}\right)^{\frac{1}{2}} \cos\phi \qquad (10)$$

Here for $\upsilon_r$ a value can be taken which equals roughly the expected one. For low energy cyclotrons one may take $\upsilon_r = 1$. For a simple cylindrically symmetric field this transformation yields for the second degree part of the hamiltonian: $K_2 = \upsilon_r I$ with the equations of motion $\dot{I} = 0$ and $\dot{\phi} = -\upsilon_r$.
Taking a new angle variable $\bar{\phi} = \phi + \theta$ and keeping the same action variable the motion is considered with respect to a comoving coordinate system. For the cylindrically symmetric field this then yields:

$$K_2 = (\upsilon_r - 1)\ I \text{ and } \dot{\bar{\phi}} = -(\upsilon_r - 1)$$

This shows the rotation of the orbit centre which moves opposite to the particle motion if $\upsilon_r > 1$. With the introduction of the action and angle variables the hamiltonian is now ready for transformations which remove the $\theta$-dependency [4]. These transformations are generally quite complicated. We only give the final result up to the fourth degree and first order in the field quantities. Further the most important terms in each degree are kept. Then

$$K = \frac{1}{2}(A_1\cos\phi + B_1 \sin\phi)(2I)^{\frac{1}{2}} + (\upsilon_r - 1 + \frac{1}{2}A_0' + (\frac{1}{2}A_2 + \frac{1}{4}A_2')\cos 2\phi + (\frac{1}{2}B_2 + \frac{1}{4}B_2')\sin 2\phi)\ I + \frac{1}{48}(D_1\cos 3\phi + D_2 \sin 3\phi)(2I)^{3/2} + \frac{1}{16}(E_0 + E_1\cos 4\phi + E_2 \sin 4\phi)\ I^2 + \ldots$$

$$D_1 = 3A_3 + 5A_3' + A_3'',\ D_2 = 3B_3 + 5B_3' + B_3'',\ E_0 = \mu'' + \mu'''$$

$$E_1 = \frac{1}{5}A_4 + \frac{19}{8}A_4' + \frac{3}{2}A_4'' + \frac{1}{6}A_4''',\ E_2 = \frac{1}{5}B_4 + \frac{19}{8}B_4' + \frac{3}{2}B_4'' + \frac{1}{6}B_4'''$$

For drawing flow lines in phase space the action and angle variables are transformed to Cartesian coordinates which represent the orbit centre coordinates (see e.g.[7]).

$$X = r_0\ (2I)^{\frac{1}{2}}\cos\phi \quad Y = r_0\ (2I)^{\frac{1}{2}}\sin\phi \qquad (11)$$

For the simple cylindrically symmetric field and scaling with $r_0$ this yields

$$K_2 = \frac{1}{2}(\upsilon_r - 1)\ (X^2 + Y^2) \qquad (12)$$

This thus gives the circular motion of the orbit centre with frequency $\upsilon_r - 1$. The hamiltonians for the complicated motion in perturbed AVF fields all have equation (12) as a central part.

*A first harmonic perturbation*. The hamiltonian up to second degree with a first harmonic perturbation is given by

$$K = \frac{1}{2}(A_1 r_0 X + B_1 r_0 Y) + \frac{1}{2}(\upsilon_r - 1)\ (X^2 + Y^2) \qquad (13)$$

The radial oscillation frequency may be taken from either numerical calculations or from analytical expressions [2,4]. The flow lines are circles around a shifted centre point:

$$K = \frac{1}{2}(\upsilon_r - 1)\left(\left(X + \frac{A_1 r_0}{2(\upsilon_r - 1)}\right)^2 + \left(Y + \frac{B_1 r_0}{2(\upsilon_r - 1)}\right)^2\right)$$

If $\upsilon_r > 1$ the centre point is shifted away opposite to the direction of the first harmonic perturbation $(A_1, B_1)$, if $\upsilon_r < 1$ the centre point is "attracted" by the first harmonic. The size of the shift is given by $\underline{\Delta x}$ :

$$|\underline{\Delta x}| = \frac{C_1 r_0}{2|\upsilon_r - 1|}$$

where $C_1$ is the amplitude of the first harmonic perturbation.

In fig. 1 and 2 two examples are given for flow lines in a threefold symmetric field with and without a first harmonic perturbation. They show the well known triangular shape [2,4] for $C_1 = o$ and the asymmetry for $C_1 \neq o$. The used values for $D$, $C_1$, $\upsilon_r - 1$ are quite normal in cyclotrons.

*A second harmonic perturbation*. A second harmonic perturbation leads to elliptical or hyperbolic flow lines in phase space:

$$K = \frac{1}{2}(\upsilon_r - 1)(X^2 + Y^2) + \frac{1}{2}(\frac{1}{2}A_2 + \frac{1}{4}A_2')(X^2 - Y^2) + (\frac{1}{2}B_2 + \frac{1}{4}B_2')\ XY \qquad (14)$$

Choosing the azimuthal zero point such that $1/2\ B_2 + 1/4\ B_2' = 0$ yields

$$K = \frac{1}{2}\left((\upsilon_r - 1) + \frac{1}{2}A_2 + \frac{1}{4}A_2'\right)X^2 + \frac{1}{2}\left((\upsilon_r - 1) - \frac{1}{2}A_2 - \frac{1}{4}A_2'\right)Y^2$$

One immediately observes the condition for stability: $\upsilon_r > 1$, $1/2A_2 + 1/4A_2' < \upsilon_r - 1$.

## 5) Smooth acceleration

The coefficients $\alpha_{1k}$ in eq (7) and (8) will slowly change during acceleration. If the accelerating structure is neglected then the equations (8) can be rewritten to simulate this slow change of the parameters. All quantities are given as functions of radius as a result of field measurements or as a result of numerical calculations. Therefore (8) is transformed by taking the radius as independent variable instead of the azimuth[7] : $\theta = 2\pi n \quad n = Cr^2$ where $n$ is the number of revolutions and $C$ a constant depending on the Dee voltage and the mean magnetic induction. It follows that

$$\dot{X} = \frac{dX}{d\theta} = \frac{1}{4\pi Cr}\frac{dX}{dr} = \frac{\partial K}{\partial Y}$$

The equations of motion then become

$$\frac{dX}{dr} = 4\pi Cr\frac{\partial K}{\partial Y},\quad \frac{dY}{dr} = -4\pi Cr\frac{\partial K}{\partial X} \qquad (15)$$

Though these equations are only approximations. They serve well in demonstrating the orbit behaviour during acceleration through radially dependent perturbations or through resonances.
In fig. 3 an example of the motion of the orbit centre in the extraction region through $\upsilon_r = 1$ is given.

This type of calculation is performed by Corsten [20] to estimate the influence of a bevelled Dee. In the Groningen cyclotron this same influence has been studied by Van Asselt together with its RF-phase dependent importance [21].

In the extraction region quite often the $\nu_r = 2\nu_z$ resonance occurs rather immediately after the $\nu_r = 1$ resonance. The hamiltonian for describing these resonances together is constructed from the hamiltonians given in section 4 with the addition of coupling terms [22]:

$$K = \frac{1}{2}(A_1 rX + B_1 rY) + \frac{1}{2}(X^2 + Y^2)(\nu_r - 1) + \frac{1}{2}(P_z^2 + Z^2)(\nu_z - \frac{1}{2}) - g'' \frac{X}{r}(Z^2 - P_z^2) - 2g'' YZP_z \qquad (16)$$

with $g'' = \frac{1}{4}(\mu' + \mu'') + \frac{1}{16} \simeq \frac{1}{4}\mu''$ in fringing fields.

In this hamiltonian X, Y, Z, $P_z$ are expressed in the same units as the radius r (e.g. in mm). If required the terms representing the main field harmonics can just be added. The axial momentum $P_z$ equals the axial angle deviation multiplied with radius and divided by the axial oscillation frequency to get circular flow lines in phase space. The hamiltonian is derived from eq. (4) using action and angle variables [22] and then transforming to $P_z$, Z, X, Y with a transformation of the type (11).

In fig.4a and b two examples of results derived from eq (16) are shown. In these cases no first harmonic was used. The data for $\nu_r$, $\nu_z$ and g" are given in fig. 6. One clearly observes the increase or decrease of the axial oscillation energy as a function of radius in dependence of the radial starting values. The total energy in both oscillators remains constant during the calculation as should be expected. In the figures the values of $(X^2 + Y^2)^{1/2}$ and $(P_z^2 + Z^2)^{1/2}$ are plotted. For the calculation of the energies one has to multiply $(X^2 + Y^2)$ and $(P_z^2 + Z^2)$ by the respective oscillation frequencies $\nu_r$ and $\nu_z$. The oscillatory behaviour after r = 500 mm has to be ascribed to the fact that for a given total energy the "central" fixed point does not coincide with $(X\neq 0, Y\neq 0, Z=0, P_z=0)$. In fig. 5 a few examples are given for a situation in which a small first harmonic field perturbation is present. The amplitude of the first harmonic is $2 10^{-4}$, its orientation is taken in four directions. One observes the rapid increase of the radial oscillation amplitude and the relatively large influence on the axial amplitude. In this case the total energy of both oscillators increases.

## 6) Acceleration with Dee structure

The geometrical structure of the RF-system is very important in the central region. Effects at large radii have to be considered when resonances are present. An example is the gap crossing resonance [5]. Another example is the influence of the position of cavities in a ring accelerator [23].

In the foregoing sections we started with a description in cylindrical coordinates and ended with hamiltonians for the orbit centre coordinates in Cartesian coordinates. The question may arise whether the use of Cartesian coordinates right from the beginning could be profitable. It turns out that for the treatment of the acceleration in the central region this works nicely [8].

However, for the incorporation of azimuthally varying magnetic fields the analytical calculations become complicated. The way of thinking may be that firstly the hamiltonian describing the influence on the centre motion due to acceleration is constructed using Cartesian coordinates and then secondly the results of the preceding sections are added to the hamiltonian. This is allowed as long as first order effects are considered. The hamiltonian including the electric fields arising from the RF-structure is given by

$$H = \frac{1}{2m}(\underline{p} - Ze\underline{A})^2 + ZeV(\theta)\cos\omega_{RF}t \qquad (17)$$

Note that the hamiltonian is time dependent. Therefore taking $-p_i$ as a new hamiltonian with the conjugate coordinate $q_i$ as independent variable does not help so much as in the time independent case. Further if the hamiltonian is expanded in cylindrical coordinates one finds essential terms with $r^{-1}$ as a factor which, yield difficulties in the required series expansions. A treatment with Cartesian coordinates leads to a way out [8,24]. One can construct direct transformations to orbit centre coordinates as well as energy and central positions phase coordinates. In a cyclotron with one Dee and an $h^{th}$ harmonic acceleration mode the final hamiltonian is [8].

$$H = f(E) + \frac{ZeV}{\pi}\frac{\sin h\phi}{h} + \frac{1}{2}(\nu_r - 1)(X^2 + Y^2) - \frac{ZeV}{2\pi}\sin(h\phi)\frac{hY^2}{2E} \qquad (18)$$

The energy is represented as $E = 1/2R^2$ with R the radius of the equilibrium orbit of the particle with energy $E_p$, the quantity V is related to the Dee voltage by $V = V(\mathrm{Dee})R^2_{max}/2E_{pmax}$ with $R_{max}$ and $E_{pmax}$ the maximum values of radius and particle energy. The charge of the particle is Ze. The radial oscillation frequency may be taken from analytical or numerical results. The function f(E) describes the isochronism of the field for centered particles: $\phi = \partial H/\partial E \simeq \partial f/\phi E = \Delta B/B$. The time is scaled with $\bar{t} = t\omega_{RF}/h$, with h the mode number of the acceleration. The quantity $\Delta B/B$ is a function depending on $R^2$ and thus on the energy E. This quantity may be known from measurements, analytical or numerical calculations, or may be constructed by hand to investigate the influence of non isochronism on the orbit behaviour. In the hamiltonian (18) the width of the accelerating gap is taken equal to zero. For a study of phase compression this may be an approximation which cannot be allowed. More extended expressions which take the Dee gap width into account follow directly from the general treatment as has been shown by W. Schulte [8].

One clearly observes in eq. (18) the coupling between the orbit centre motion and the longitudinal motion. This coupling becomes large for off centered particles, high harmonic numbers and small energies. The coupling may give rise to an unstable orbit centre motion in the same way as a second harmonic perturbation in the magnetic field can do it. For studying the combined effects of Dee structure and first and second harmonic perturbations one simply may add the relevant parts of the hamiltonians (13) and (14) to (18):

$$\Delta H_{1,2} = \frac{1}{2}(A_1 rX + B_1 rY) + \frac{1}{2}(\frac{1}{2}A_2 + \frac{1}{4}A_2')(X^2 - Y^2) + (\frac{1}{2}B_2 + \frac{1}{4}B_2')\,XY$$

For a study of a cyclotron with two small extra Dees for flattopping the hamiltonian for a two Dee system driven on the $3h^{th}$ harmonic must be built in.

The extra terms are found from a coordinate rotation applied on the two-Dee hamiltonian given by Schulte [8].

$$\Delta H_{3h} = \frac{ZeV_3}{\pi}\, 2\frac{\sin 3h\alpha}{3h} \sin 3h(\phi-\phi_3) + \frac{C}{2E}\,(x\cos\beta - y\sin\beta)(x\sin\beta + y\cos\beta)\cos 3h(\phi-\phi_3) + \frac{D}{2E}\left(\frac{1}{2}(x\sin\beta + y\cdot\cos\beta)^2\cos^2\alpha + \frac{1}{2}(x\cos\beta - y\sin\beta)^2\sin^2\alpha\right)\cdot\sin 3h(\phi-\phi_3)$$

Here $V_3$ corresponds to the voltage on the extra Dees, $\beta$ is the rotation of the Dees with respect to the acceleration gap of the main Dee, $\alpha$ is the half angle of the extra Dees, $\phi_3$ is the RF-phase. Further C =

$$C = \frac{2ZeV_3\cdot}{\pi}(3h)\cos 3h\alpha\cdot\cos\alpha\cdot\sin\alpha,\quad D = \frac{-2ZeV_3\cdot}{\pi}(3h)\sin 3h\alpha$$

The examples given are only demonstrations to show how to construct the final hamiltonian model. Coupling effects in multi Dee systems may lead to a serious elongation of the phase space figures resulting finally in bad beam quality due to RF-phase mixing. These effects will not be treated here further except for the illustration in fig. 7. In this figure the influence of the azimuthal extension of the Dees in a two Dee system on the phase space behaviour of the orbit centres is shown.

## 7) Acceleration with RF cavities

For the study of orbit dynamics at large radii the use of curvilinear or cylindrical coordinates is to be preferred to Cartesian coordinates as has been done in the preceding section. The closed orbit has a circle like shape and deviations can be expressed as small quantities relative to this orbit. Also field quantities can rather easily be represented by fourier series. The acceleration in cavities or in dee gaps in fact originates from the RF magnetic field. Therefore contrary to the ad hoc addition of a potential function, the RF fields are now incorporated in the magnetic vector potential [9,10]. The relativistic hamiltonian is given by

$$H = \left(E_r^2 + (p_x - eA_x)^2\cdot c^2 + (p_z - eA_z)^2\cdot c^2 + \left(\frac{p_s}{1+\frac{x}{\rho(s)}} - eA_s\right)^2\cdot c^2\right)^{\frac{1}{2}} \quad (19)$$

$E_r$ is the rest energy of the particle, c the velocity of light, $\rho$ the radius of curvature, s the position along the orbit. To get rid of the square root a series expansion should be made. However, we only know that the quantities $p_x$, $p_z$, x, z are small. The longitudinal momentum $p_s$ may be quite large. It is therefore convenient to introduce a new longitudinal momentum which represents the deviation of the old momentum from a prescribed function. The choice of this function leads to a central orbit.

The further development of the hamiltonian can lead to a combined description of transversal and longitudinal motion under the influence of RF acceleration and betatron acceleration including coupling effects.

*Betatron motion.* For a cylindrically symmetric field in a betatron the hamiltonian does not depend on the longitudinal coordinate s. Therefore the conjugate momentum $p_s$ is a constant of motion. If this constant is taken equal to zero one observes that in the expansion of the square root no first degree terms are present.

We then have a description around an equilibrium orbit: $p^2_{kin} = e^2A_s^2$ and we know $p_{kin} = -eBr_0$. Therefore $A_s = -Br_0$. This means that the enclosed flux has an average magnetic induction which is twice the value at the orbit radius (the so-called betatron condition). A more rigorous treatment is given in ref [10].

*Synchrotron motion.* For the synchrotron motion the vector potential is split into a slowly varying part ($A_{s,s}$) and a fast oscillating part ($A_{s,f}$) due to the RF fields. A new longitudinal momentum is now defined which describes the deviation of the old momentum from a slowly changing prescribed function added with the fast oscillating part of the vector potential $\bar{p}_s = p_s - p_{so}(t) - eA_{s,f}(s,t)$. The hamiltonian (19) becomes

$$H = \left(E_r^2 + p_x^2c^2 + p_z^2c^2 + \left(\frac{\bar{p}_s + p_{so}}{1+\frac{x}{\rho}} - eA_{s,s}\right)^2 c^2\right)^{\frac{1}{2}} + s\frac{d}{dt}p_{so}(t) - e\int^s E_s(s,t)\,ds \quad (20)$$

with $E_s$ the electric field strength in the accelerating gaps. The influence of the betatron acceleration is incorporated in $A_{s,s}$ and $P_{so}$, the influence of the accelerating structure in the last term. In the case of a Dee structure this term is a periodic function in s and equals the potential function that has been used earlier in section 6:

$$-e\int^s E\,(s,t)\,ds \rightarrow eV(s)\cos\int\omega_{HF}(t)\,dt \quad \text{(Dee)}$$

In case of a cavity one acquires a step function in s:

$$-e\int^s E\,(s,t)\,ds \rightarrow -\int^s E_s(s)\,ds\cdot\cos\int\omega_{HF}\,dt \quad \text{(cavity)}$$

If radial electric field strenghts are important for the orbit behaviour they should be taken into account right here before going further.

After expanding the square root one observes in second degree a coupling between $\bar{p}_s$ and x. The removal of this coupling term leads to a general definition of the central position phase in time dependent fields [10]. Scaling of the parameters, removal of first degree parts and defining a phase coordinate with respect to the RF-phase leads to a description of an accelerated equilibrium orbit, the synchrotron oscillations and coupling phenomena between transversal and longitudinal modes of oscillation. The results are equivalent at least to those found from the methods given in section 5, but give a wider range of applicability [10,23].

## 8) Concluding remarks

The analytical theory for the acceleration of single particles has been developped until now in such a way that the results are reliable for a first study of orbit dynamics in e.g. central regions, extraction regions and resonances. The use of resulting hamiltonians is generally simple and the meaning of the relevant coordinates and momenta clear. The construction of the hamiltonians is sometimes easy. Often, however, it is an art which asks for some experience. As a start for the study of orbit theory one should first consult the references [19] and [25]. Extended information on orbit theory can be found in the Proceedings of the International Conferences on cyclotrons.

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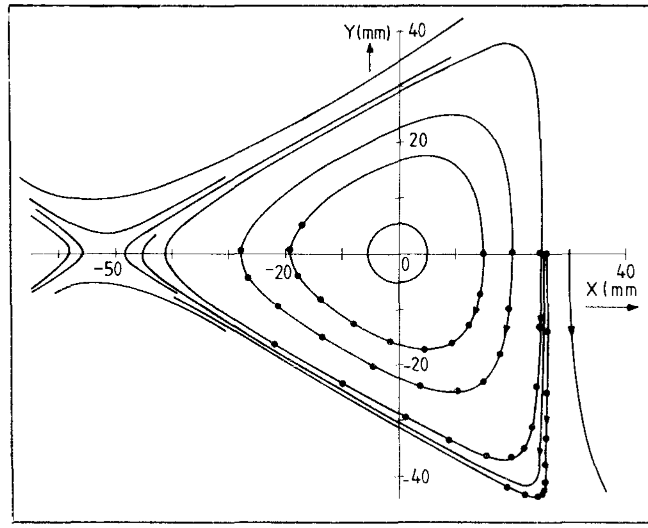


fig. 1. The flow lines in a 3-fold symmetric magnetic field (see also ref.3 and 4) without a first harmonic. The points indicate a progress of 3 revolutions. $D_1 = 3$, $D_2 = 0$, $\nu_r - 1 = 0.02$

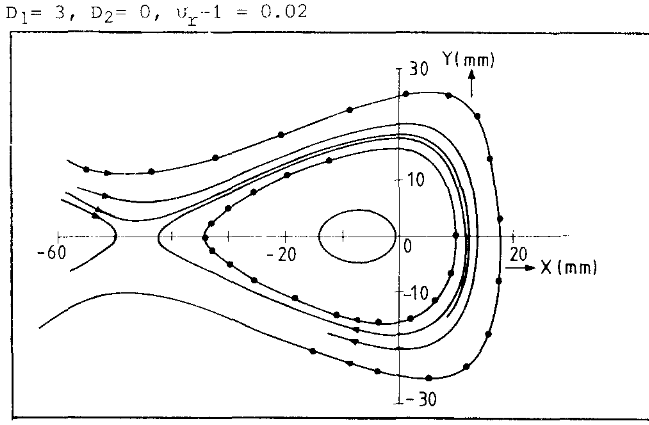


fig. 2. The flow lines in a 3-fold symmetric field with a first harmonic $A_1 = 0.0005$, $B_1 = 0$, other data as used for fig. 1. One clearly observes the shift of the closed orbit and the decrease of the stable region.

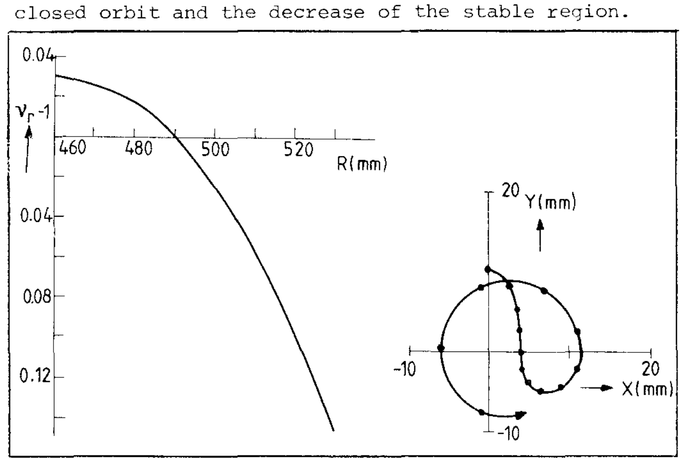


fig. 3a. An example of the motion of the orbit centre in the extraction region. The radial interval is $R_i$ = 480 mm to $R_f$ = 520 mm. $D_1 = 3$, $D_2 = 0$, $A_1 = 0.0005$, $B_1 = 0$, $\nu_r = 1$ is given in fig. 3b. The points indicate a progress of 3 revolutions. The last point corresponds to R = 513.6 mm. In extraction studies the radius should be added to the X coordinate if the extractor entrance lies in the X-direction. The cycloidal path in $r$-$p_r$ phase space then follows.

fig. 3b. The radial oscillation frequency as used for the example in fig. 3a.

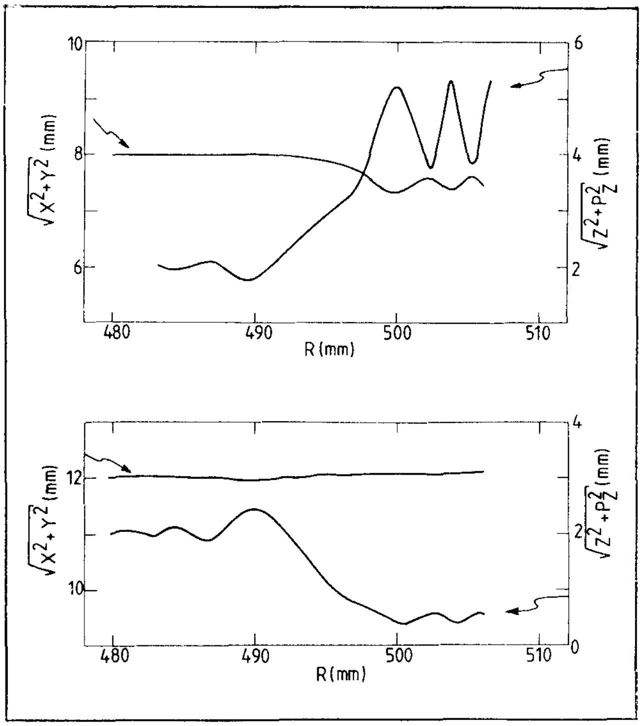

fig. 4a. The radial and axial oscillation amplitudes as a function of radius without a first harmonic perturbation, showing the influence of the $\upsilon_r=2\upsilon_z$ resonance. The data for $\upsilon_r$, $\upsilon_z$ and g" are given in fig.6. In fig. 4a the start values are at R = 480 mm: X = 8 mm, Y = 0, Z = 2 mm, $P_z$ = 0.
In fig. 4b the start values are: X = -12 mm, Y = 0, Z = 2, $P_z$ = 0. In the interval R = 480 - 510 mm thirty revolutions were taken.

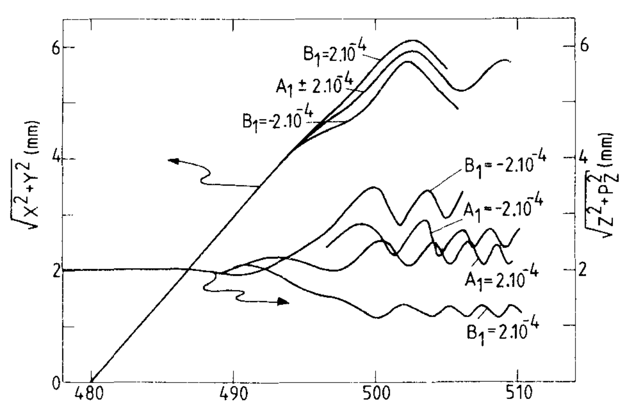

fig. 5. The radial and axial oscillation amplitudes as a function of radius with a small first harmonic perturbation for four different cases showing the combined influence of the $\upsilon_R$ = 1 and $\upsilon_R = 2\upsilon_z$ resonances: $A_1$ = 0.0002, $B_1$ = 0/ $A_1$ = -0.0002, $B_1$ = 0/ $A_1$ = 0, $B_1$ = 0.0002/ $A_1$ = 0, $B_1$ = -0.0002. The data for $\upsilon_r$, $\upsilon_z$ and g" are given in fig. 7. In the interval R = 480 - 510 mm thirty revolutions were taken. The start values at R = 480 mm are X = 0, Y = 0, Z = 2 mm, $P_z$ = 0.

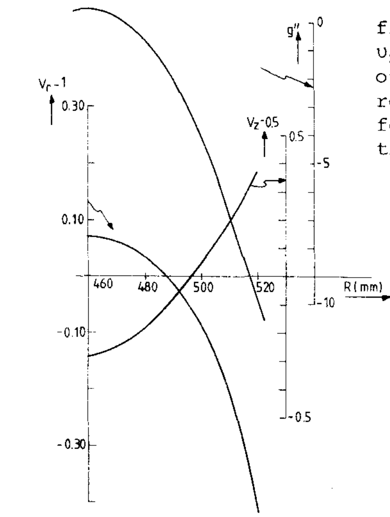

fig 6. The values of $\upsilon_r$, $\upsilon_z$ and g" as a function of radius. The values are roughly representative for the Eindhoven cyclotron.

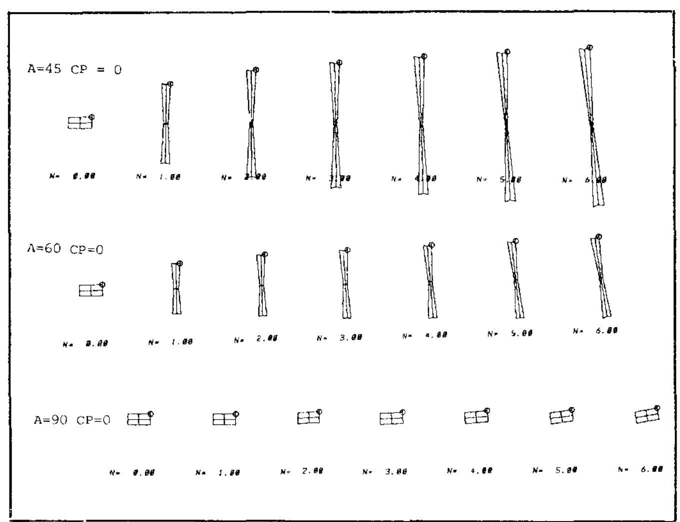

fig. 7. Radial phase space figures during the first six revolutions in a minicyclotron in construction at the University of Eindhoven, showing the influence of the azimuthal Dee width. Initial energy 25 keV, energy gain 100 keV per revolution; central position phase at the start = $0^o$; size of the initial area 1 x 2 $mm^2$; second harmonic mode of operation; Dee width 45, 60 and 90. It is clearly shown that for good beam quality one has to adapt the ion source emittance; if sufficient beam current is available, one has to cut away unwanted parts of the area by means of central diaphragms if the Dee width does not equal $\pi/h$, with h the harmonic mode number.

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