AMPLITUDE GROWTH FROM THE RAPID TRAVERSAL OF A HALF-INTEGER RESONANCE

R. Baartman, G.H. Mackenzie
TRIUMF, 4004 Wesbrook Mall, Vancouver, B.C., Canada V6T 2A3
M.M. Gordon
Michigan State University, East Lansing, MI 48824

Summary

Traversal of a half integer resonance (ν=n/2) in the presence of an nth harmonic component of the field index $(\overline{R}/\overline{B})(dB_n/dr)$ increases the incoherent betatron amplitude of the circulating beam. The amplitude growth is usually estimated by considering motion only within the stop band. Studies using our general orbit code GOBLIN have shown that this approach under-estimates the effect of the resonance in cases where the amplitude growth is <100%. Most of the amplitude gain in these cases is acquired outside the stop band where the envelope function changes so rapidly that the motion is no longer adiabatic. Solving the differential equation for this fast passage case yields an expression for amplitude gain which is found to be in excellent agreement with our GOBLIN studies. The stop band may be traversed rapidly either because of a large energy gain per turn, a weak perturbing field, or both. It is possible to apply this method to rapid traversal of other resonances. Our linear motion code COMA confirms a correlation between betatron amplitude acquired and the energy spread in the TRIUMF extracted beam. A recent experiment is in agreement with the new expression for amplitude growth.

Introduction

When the magnetic field tolerances of a 3.5 GeV, and 8-15 GeV superconducting cyclotrons were being investigated it was found that the stretching of the beam ellipse while traversing a ν_x = n/2 resonance and the subsequence mis-match to the cyclotron acceptance were typically 10 times larger than predicted by analytic calculations.[1] The "realistic" cases being studied were for fairly weak driving terms of order $(\overline{R}/\overline{B})(dB_n/dr)$ = 0.1 and for a large energy gain of 20 MeV per turn. The particles being simulated in our general orbit code GOBLIN spent one turn or less in the stop band.

The standard analysis of resonance crossing assumes that the beam makes several turns in the resonance and predicts that the amplitude gain will vary as the square of the driving term, since both the strength of the resonance and the number of turns spent in the stop band are proportional to this term. The GOBLIN studies showed that in fact the amplitude gain varied linearly with the driving term and moreover most of the amplitude was gained outside the stop band. This amplitude growth is due mostly to the rapid non-adiabatic change in the cyclotron envelope function β near the resonance.

The situation is summarized in Fig. 1 where the beta function and the amplitude gain $A/A_{initial}$ are plotted as a function of energy through a half integer resonance. The amplitude growth was determined at each energy by finding the size of the acceptance ellipse, calculated in the absence of the field perturbations, which would just contain the stretched beam ellipse. The field perturbation slightly alters the shape of the matched ellipse far from the resonance and this accounts for the fact that $\ln(A/A_i) \neq 0$ at the start and does not have a flat plateau at the end. These studies were made with an azimuthally homogeneous isochronous field, $\overline{B} = \gamma B_c$.

A re-examination of the TRIUMF field showed that a similar situation applied. Although the energy gain

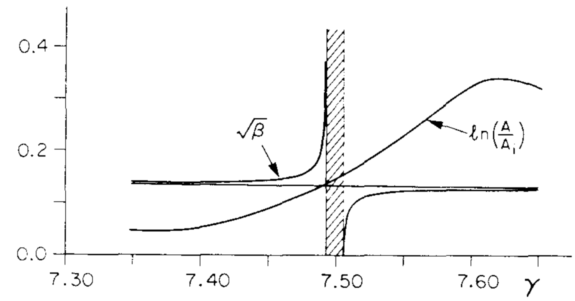

Fig. 1. Calculated β-function and amplitude gain through the ν_r = 15/2 resonance with an energy gain of 45 MeV/turn. The shaded area is the stop band.

is much less, typically 0.3 MeV per turn, the field shimming had intentionally minimized the residual third harmonic gradient in the vicinity of ν_x = 1.5, and thus the natural stop band was quite narrow, only four turns. GOBLIN calculations showed that for this case too the amplitude growth was greater than predicted analytically.

The third harmonic gradient can be varied by means of harmonic coils and this suggested that the results of the calculations may be able to be checked experimentally.

Theory

At a half integer resonance, $\nu_{x,y}$ = n/2, stability is lost if there is a first radial derivative of the nth harmonic of the magnetic field. Consider radial motion. This can be described by the differential equation

$$\frac{d^2x}{d\theta^2} + \nu^2 x = bx\cos(n\theta+\alpha) \qquad (1)$$

where $b = \overline{R}/\overline{B} \cdot dB/dR|_{nth\ harmonic}$. One can show[2] that the driving field changes ν to ν^* given by

$$(\nu^*-n/2)^2 = (\nu-n/2)^2 - \left(\frac{b}{2n}\right)^2 \qquad (2)$$

At ν=n/2, ν^* has an imaginary part Im(ν^*) = b/2n. Since $x \sim e^{i\nu\theta}$, this implies the possibility of an exponentially growing betatron amplitude. Also, ν^* has an imaginary part over a range $\Delta\nu$ = b/n around ν=n/2. Using the rate of change of ν, we can therefore show that if we write $x = Ae^{i\nu\theta}$ then the amplitude growth $A_{final}/A_{initial}$ is given by

$$\ln(A_f/A_i) = \left(\frac{\pi}{2n}\right)^2 \frac{b^2}{\nu_t} \qquad (3)$$

where ν_t is the change in ν per turn.

This is the common formula used, for example, to study synchrotron instabilities or half integer resonance extraction. This is called the adiabatic theory because it assumes implicitly that ν changes slowly enough that one can define an effective ν^* at every point of the orbit.

The theory applicable to a fast resonance passage is as follows. Equation 1 is solved using action-angle variables. Let $x = A/\sqrt{\nu}\cos\phi$ and $x' = -\sqrt{\nu}A\sin\phi$. Then

$$\phi' = \nu - \frac{b}{\nu} \cos^2 \phi \cos(n\theta+\alpha) \tag{4}$$

$$\frac{A'}{A} = - \frac{b}{2\nu} \sin(2\phi)\cos(n\theta+\alpha) \tag{5}$$

To first order, these can be written

$$\frac{A'}{A} = - \frac{b}{2\nu} \sin(2\int \nu d\theta)\cos(n\theta+\alpha) \tag{6}$$

and integrated to yield

$$\ell n(A_f/A_i) = \frac{\pi}{\sqrt{2n}} \frac{b}{\sqrt{\nu_t}} \tag{7}$$

Since this is just the square root of twice the adiabatic formula, we see that resonance growth becomes more important than non-adiabatic growth when $\ell n(A_f/A_i) > 2$.

In fact, the canonical action variable is not A but A^2. Furthermore, we have, for the sake of clarity, ignored adiabatic damping.

From a closer inspection of eq. (4), one can show that most of the amplitude gain takes place inside a band of width $\Delta\nu = \sqrt{\nu_t}$ about the resonance $\nu = n/2$. This is to be compared with the stop-band width of $\Delta\nu = b/n$. The condition for being in the non-adiabatic regime is therefore $b/n << \sqrt{\nu_t}$, and this is equivalent to the condition $\ell n(A_f/A_i) << 2$.

The agreement between the non-abiabatic theory, eq. (7), and orbit calculations is shown in Tables I and II. In Table I, we have summarized results of a study of the $\nu_r = 10/2$ resonance at 4.22 GeV in a 30 sector, 8.5 GeV cyclotron. In Table II, we compare the general orbit calculations for the $\nu_r = 3/2$ resonance in TRIUMF with the theoretical expression. The GOBLIN results agree with products of eq. (4).

Table I

$\ell n\ (A_{final}/A_{initial})$ for the ν_r=10/2 resonance in an 8.25 GeV cyclotron

$dB/dR\|_{10}$	GOBLIN	Eq. (3)	Eq. (7)
0.000	0.00	0.000	0
0.010T/m	0.13	0.012	0.15
0.020T/m	0.26	0.048	0.30
0.040T/m	0.59	0.192	0.60

Table II

$\ell n(A_f/A_i)$ for the $\nu_r = 3/2$ resonance in TRIUMF

$dB/dR\|_3$	GOBLIN	Eq. (3)	Eq. (7)
0.20G/in	0.53	0.10	0.45
0.40G/in	1.06	0.40	0.90

In the transition region around $\ell n(A_f/A_i) = 2$ both formulas (3) and (7) underestimate the amplitude gain. To explore this regime, eqs. (4) and (5) were solved numerically. The results for the case $\nu_t/n^2 = 10^{-4}$ are plotted in Fig. 2. One can clearly see the validity of both formulas (3) and (7) (dashed lines) in their particular regimes.

Possibility of Experimental Confirmation

The phenomenon being discussed is linear. The ellipse is stretched but not distorted. As the beam is accelerated away from the resonance the ellipse occupied by the beam, which is now mis-matched to the cyclotron acceptance, begins to rotate (Fig. 3). The rate of precession increases as $(\nu_x - n/2)$ increases. At present, the TRIUMF cyclotron cannot be operated in a separated turn mode at $\nu_x = 3/2$ (428 MeV). The

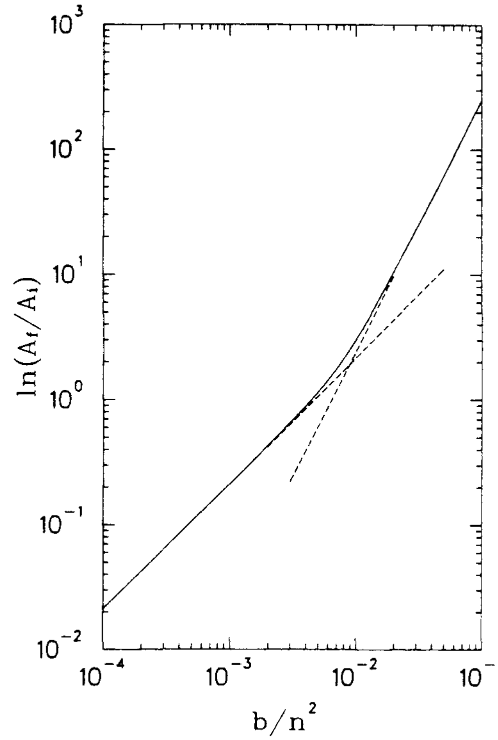

Fig. 2. Amplitude gain calculated by integrating the action-angle equations with $\nu_t/n^2 = 10^{-4}$.

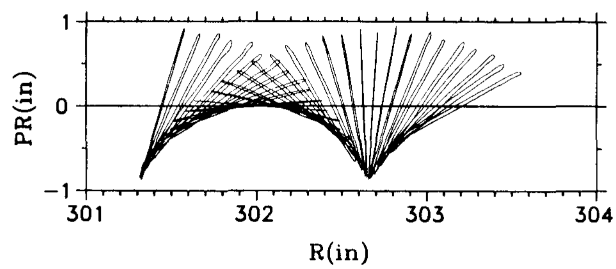

Fig. 3. Turn-by-turn development of the beam in radial phase space calculated for passage through the ν_r = 3/2 resonance in TRIUMF with dB_3/dR = 1.0 G/in.

amplitude gain from passage through this resonance can therefore not be measured directly. However, it may be inferred from measurements of the properties of the extracted beam.

A carbon wire stripping foil 33 μ dia. extracts about 2% of the circulating beam and may be said to sample the beam but not disturb it. From Fig. 3 it can be seen that the energy spread of the extracted beam is narrow when the beam ellipse is upright and wider when the major axis is radial. The extracted energy spread will vary with extraction energy or radius in a similar manner to the modulation of the turn width. Eventually the various phases making up a beam of finite phase width will make a different number of turns to a given radius or energy and this precessional mixing will smear out the modulation.

The TRIUMF six sector cyclotron has a series of coils of azimuthal width 60° and radial width ~0.7 m centered around each pole. These harmonic coils may be connected in such a way as to provide a controllable change in the third harmonic amplitude and gradient of the magnetic field, however, only two phases are possible, separated by 180° which we call mode A and mode B. In general neither of these two phases will coincide with the residual third harmonic phase.

Monte Carlo Simulations

These calculations were performed to determine the degree of correlation between the energy spread of the extracted beam and the change in incoherent amplitude A_f/A_i. They were also necessary to determine the experimental precision required. The linear motion code COMA[3] was used. This tracks particles using matrices from the equilibrium orbit code CYCLOP and can describe the interaction of the beam with a slit, probe or extraction foil.

Files of transfer matrices were obtained for the base magnetic field, which includes the residual third harmonic, and for the base field plus the additional contributions due to harmonic coils powered at several strengths for the two possible phases. It is known experimentally that TRIUMF can select a beam for acceleration ±2° in phase width although this may be broadened to ±4° say at 430 MeV due to field or RF instabilities. COMA simulations started at 410 MeV, below the $\nu_x = 3/2$ resonance at 428 MeV. A beam ellipse matched to the cyclotron acceptance and ±3° wide was populated and accelerated through the resonance. Results from one of these sets of calculations is shown in Fig. 3. The foil in the simulations was wider than that used experimentally in order to provide adequate statistics. It was found that at some energies the extracted energy spread was narrower than that of the incoming beam. These energies correspond to the stretched ellipse having a narrow radial projection.

Experimental Procedure

Measurement of Energy Width

The population distribution in the energy of the extracted beam is measured using a beam profile monitor at a dispersed focus at target 4BT1 in BL4B. A schematic layout of this section of the beam line is given in Ref. 4. The transfer matrix terms R_{16} and R_{12} were measured between the extraction foil and the profile monitor by using harmonic coils in a first harmonic mode to alter the extraction energy and the direction by a known amount.[5] The computed tune was intended to give R_{12}=0, measured 0.1, and R_{16}= 3 cm per % $\Delta p/p$, measured 2.9. R_{11} was not measured but computed to be 2, the object size of course is small unless the wire foil is bent.

Beam Preparation

The injected phase width was reduced to ±2° by means of a series of flags and slits. The radial coherent oscillations were reduced to <0.5 mm by means of coils compensating for first harmonic effects. The vertical coherent amplitude was reduced to <1 mm by means of electrostatic plates operating on the first few turns. The slits and a vertical flag constrained the radial incoherent amplitude to 1 mm and the vertical amplitude to 2 or 3 mm. These properties were checked at 70 MeV, beyond the inner non-adiabatic region, by means of differential and radial finger probes. In principle the vertical properties should not affect the measurement, however, in practice we are aware of regions of (x,y) coupling. Also a narrow vertical width makes tuning easier and insures that the foil makes a complete sample of the beam.

Results

Measurements were made between 420 and 450 MeV, with harmonic coil #13 giving a third harmonic gradient imperfection of 1.0 G/in in both of the possible phases and with the harmonic coil off. Some of the results are given in Fig. 4. The profile monitor resolution limited the minimum measurable energy spread to

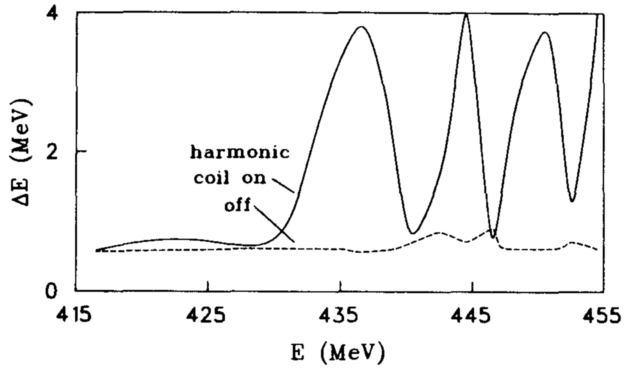

Fig. 4. Measured energy spread (FWHM) as a function of energy for dB_3/dR = 1.0 G/in.

~500 keV. Measurements of energy spread vs. harmonic coil strength were made at 436.5 MeV. This energy corresponds to a broad stable peak in energy spread vs. energy (see Fig. 3).

Inspection of Fig. 3 shows that the best measure of radial amplitude is made at the broad peaks where about 10 turns overlap and the foil samples each with roughly similar efficiency; the total energy width is a measure of the number of turns intercepted by the foil. These calculations were made for an emittance that assumes no degradation has taken place between 70 and 410 MeV. This is true for the majority of the beam, however, the maximum energy spread is determined by the halo of unknown size but whose ellipse matches the cyclotron acceptance since considerable precessional mixing will have taken place over ~1000 turns. The ellipse size was estimated by normalizing the COMA calculated energy distribution with that measured at 463.5 MeV with 1.1 G/in.

Fig. 5 compares the base energy width measured at several field gradients with the prediction of eq. (4) and (7) and with the composite theory which integrates both effects. The residual third harmonic gradient amplitude of 0.1 G/in. and phase of 45° (third harmonic phase with respect to the harmonic coil) have been included in the determination of perturbing field strengths. It can be seen for third harmonic field gradients below 0.8 G/in that the beam quality is worse than predicted by eq. (4) and is in agreement with the

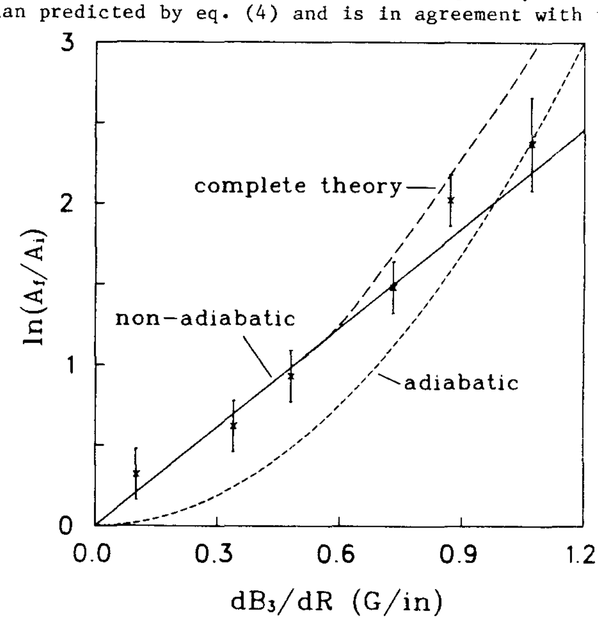

Fig. 5. Measured and calculated amplitude gain due to the ν_r = 3/2 resonance in TRIUMF vs. third harmonic gradient field strength.

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non-adiabatic theory. For clarity the data point error bars include uncertainties included in the theoretical calculations, e.g. uncertainties in the value of ν_t. Data at 1.1 G/in and perhaps 0.8 G/in should be discounted since the stretching is large and the stop band width 8.5 MeV at 0.8 G/in.) is such that not all turns sampled by the stripping foil have passed completely through the resonance.

These results were obtained after one beam shift and although the agreement is good it is hoped to repeat the experiment with a dispersion of 10 cm/%ΔP/P and take data at the smaller values of perturbing strength.

Acknowledgements

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