

DYNAMICS OF TWISS PARAMETERS FROM THE GEOMETRICAL VIEWPOINT

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Abstract

We show that with an appropriate parametrization the linear transport of the Twiss parameters can be viewed as a bilinear (or Moebius) map of the upper complex half-plane (which is the hyperbolic plane) into itself. Using then elementary techniques of hyperbolic geometry we classify transformations of the Twiss parameters into elliptic, parabolic and hyperbolic types and, for each type, present its typical phase space portraits.

INTRODUCTION

We consider the linear beam dynamics in one degree of freedom (let's say, horizontal) and, as usual, describe the transport of a charged particle beam between two beamline locations by a 2×2 real symplectic matrix, i.e. by the element of the group $\text{Sp}(2, \mathbb{R})$. As soon as one realizes that the factor group $\text{Sp}(2, \mathbb{R})/\{\pm I\}$ is isomorphic to the restricted Lorentz group of the three dimensional Minkowski space and to the group of orientation-preserving isometries of the hyperbolic plane, it is tempting to look for geometrical interpretations of the beam dynamics, i.e for the sets of the beam parameters dynamics of which realizes these isomorphisms. Because the group $\text{Sp}(2, \mathbb{R})$ is the mathematical instrument applicable to many problems, in recent years a number of geometrical interpretations have been developed in different areas of physics (see, for example, [1] and references therein), but such a geometrical viewpoint on the beam dynamics was not explored in accelerator science until the appearance of the paper [2], where we introduced it and proved its usefulness in the framework of the very practical problem of beam emittance and Twiss parameters measurements.

In this paper we continue the study started in [2] and show that the linear transport of the Twiss parameters in one degree of freedom can be viewed as a Moebius map of the upper complex half-plane (which is one of the models of the hyperbolic plane) into itself. Using then elementary techniques of hyperbolic geometry we classify transformations of the Twiss parameters into elliptic, hyperbolic and parabolic types and, for each type, present its typical phase space portraits.

We stress that the geometrical viewpoint does not offer any inherent advantage in terms of efficiency in solving practical problems; rather, we hope that this approach will complement the more standard algebraic techniques and together they will help to obtain a better physical and geometrical feeling for the dynamics of the Twiss parameters.

FROM TRANSPORT OF SECOND-ORDER BEAM MOMENTS TO HYPERBOLIC GEOMETRY OF TWISS PARAMETERS

Let us consider a collection of points in one degree of freedom phase space (a particle beam) and let, for each particle, $\mathbf{w} = (x, p)^\top$ be a vector of canonically conjugate coordinate x and momentum p . Then, as usual, the beam (covariance) matrix is defined as

$$\Sigma = (\Sigma_{km}) = \langle (\mathbf{w} - \langle \mathbf{w} \rangle) \cdot (\mathbf{w} - \langle \mathbf{w} \rangle)^\top \rangle, \quad (1)$$

where the brackets $\langle \cdot \rangle$ denote an average over a distribution of the particles in the beam. Let

$$A(s_1, s_2) = \begin{bmatrix} a_{11}(s_1, s_2) & a_{12}(s_1, s_2) \\ a_{21}(s_1, s_2) & a_{22}(s_1, s_2) \end{bmatrix} \in \text{Sp}(2, \mathbb{R}) \quad (2)$$

be a matrix which propagates particle coordinates from the state s_1 to the state s_2 , i.e let

$$\mathbf{w}(s_2) = A(s_1, s_2) \mathbf{w}(s_1). \quad (3)$$

Then from (1) and (3) it follows that the matrix Σ evolves between these two states according to the congruence

$$\Sigma(s_2) = A(s_1, s_2) \Sigma(s_1) A^\top(s_1, s_2). \quad (4)$$

Let us first extend the domain of the transformation rule (4) from positive semidefinite symmetric matrices to arbitrary symmetric matrices and then let us associate with every 2×2 symmetric matrix Σ the three component vector

$$\mathbf{m}(\Sigma) = (\Sigma_{11}, \Sigma_{12}, \Sigma_{22})^\top. \quad (5)$$

With this association the transformation law for the 2×2 symmetric matrices (4) becomes a linear transformation in the three dimensional space of \mathbf{m} vectors

$$\mathbf{m}(s_2) = T(s_1, s_2) \mathbf{m}(s_1), \quad (6)$$

where the matrix $T = T(A)$ is determined by the relation

$$T(A) = \begin{bmatrix} a_{11}^2 & 2a_{11}a_{12} & a_{12}^2 \\ a_{11}a_{21} & a_{11}a_{22} + a_{12}a_{21} & a_{22}a_{12} \\ a_{21}^2 & 2a_{21}a_{22} & a_{22}^2 \end{bmatrix}. \quad (7)$$

For an arbitrary $A \in \text{Sp}(2, \mathbb{R})$, the matrix $T(A)$ has unit determinant and all matrices T form a group (T -group) of which the symplectic group $\text{Sp}(2, \mathbb{R})$ is the double cover (the matrices $\pm A$ generate the same matrix T). Moreover, an arbitrary matrix T satisfies

$$T^\top S T = S, \quad S = \begin{bmatrix} 0 & 0 & 1/2 \\ 0 & -1 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}. \quad (8)$$

It is a remarkable fact which means that the action of the T -group on \mathbf{m} vectors preserves the symmetric bilinear form

$$B(\mathbf{m}_1, \mathbf{m}_2) = \mathbf{m}_1^\top S \mathbf{m}_2, \quad (9)$$

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which therefore defines invariant metric. Because the matrix S has two negative and one positive eigenvalues (-1, -1/2, and 1/2), this invariant metric is indefinite.¹

The emittance (the invariant norm) of a vector $\mathbf{m} = (m_1, m_2, m_3)^T$ is defined to be the complex number

$$\varepsilon(\mathbf{m}) = \sqrt{\mathbf{m}^T S \mathbf{m}} = \sqrt{m_1 m_3 - m_2^2}, \quad (11)$$

where $\varepsilon(\mathbf{m})$ is either positive, zero, or positive imaginary.

In the following we will say that the vector \mathbf{m} is beamlike, if the corresponding to it symmetric 2×2 matrix Σ is positive definite, i.e. if the first component m_1 of the vector \mathbf{m} and its emittance $\varepsilon(\mathbf{m})$ are both positive. Note that if \mathbf{m}_1 and \mathbf{m}_2 are two beamlike vectors, then

$$\mathbf{m}_1^T S \mathbf{m}_2 \geq \varepsilon(\mathbf{m}_1) \varepsilon(\mathbf{m}_2), \quad (12)$$

which is the reverse Cauchy-Bunyakovsky-Schwarz inequality. Moreover, the two sides in (12) are equal if and only if \mathbf{m}_1 and \mathbf{m}_2 are two proportional vectors.

So we have obtained the following geometric picture. The 2×2 symmetric matrices are put into one to one correspondence with the points of the three dimensional indefinite metric space, where the nondegenerate beam matrices occupy the convex region for which the nonnegative ($m_1 \geq 0$) part of the conical surface $\varepsilon^2(\mathbf{m}) = 0$ is the boundary. Under the action of the T -group this convex region splits into a set of the positive ($m_1 > 0$) sheets of the two-sheeted hyperboloids $\varepsilon^2(\mathbf{m}) = const > 0$ (orbits), and on each orbit the T -group acts transitively (see Fig.1).

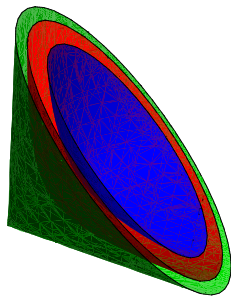


Figure 1: Boundary of the set of the beamlike vectors (green) and two invariant orbits (red and blue) inside it.

If emittance of a beamlike vector \mathbf{m} is known, then dynamics of this vector is completely determined by the behavior of its projection onto the special orbit labeled by the emittance equal to one (Twiss surface), i.e. by the dynamics of the Twiss vector \mathbf{t} and the Twiss parameters β , α , and γ :

$$\mathbf{t}(\mathbf{m}) = (\beta(\mathbf{m}), -\alpha(\mathbf{m}), \gamma(\mathbf{m}))^T \stackrel{\text{def}}{=} \mathbf{m} / \varepsilon(\mathbf{m}). \quad (13)$$

¹ If, instead of the association law (5), one uses the rule

$$\mathbf{m}(\Sigma) = ((\Sigma_{11} + \Sigma_{22}) / 2, -\Sigma_{12}, (\Sigma_{11} - \Sigma_{22}) / 2)^T, \quad (10)$$

then one obtains much more known geometry. The space of \mathbf{m} vectors becomes the three dimensional Minkowski space with the standard metric given by the matrix $S = \text{diag}(1, -1, -1)$, and the T -group turns into the restricted Lorentz group $SO^+(1,2)$. It is clear that both approaches are isomorphic, but the geometry associated with the rule (5) seems to be better suited for the beam dynamics purposes.

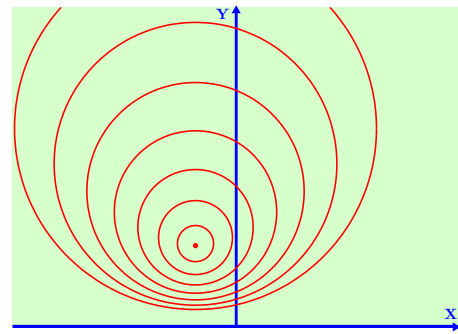


Figure 2: Flow lines of elliptic Moebius transformation.

So in the next step, let us restrict our attention to the dynamics of the Twiss parameters on the Twiss surface, alone. In this situation, the (invariant) distance function on the Twiss surface induced from the ambient metric (9) takes on the form

$$d_H(\mathbf{t}_1, \mathbf{t}_2) = \text{arccosh}(m_p(\mathbf{t}_1, \mathbf{t}_2)), \quad (14)$$

where

$$m_p(\mathbf{t}_1, \mathbf{t}_2) = \mathbf{t}_1^T S \mathbf{t}_2 = \frac{1}{2} (\beta_1 \gamma_2 - 2\alpha_1 \alpha_2 + \beta_2 \gamma_1) \quad (15)$$

is the betatron mismatch parameter; and, due to isomorphism with the forward sheet of the unit hyperboloid in the standard three dimensional Minkowski space, one can state that the Twiss surface considered together with the metric (14) constitutes a model of two-dimensional hyperbolic geometry (the so-called ‘‘hyperboloid model’’).

THE UPPER HALF-PLANE MODEL

In the previous section we have seen that the dynamics of the Twiss parameters can be visualized in the framework of the hyperboloid model of hyperbolic geometry, which does not look to be very convenient for the human eyes because one has to deal with the curved two-dimensional surface (Twiss surface) in the three-dimensional Euclidean space. Fortunately, there are other models of hyperbolic geometry, and, in this section, we will discuss visualization of two-dimensional hyperbolic geometry of Twiss parameters using the upper half-plane model [3].

The upper half-plane model is the set of complex numbers z with positive imaginary part

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\} \quad (16)$$

together with the hyperbolic distance function

$$d_H(z_1, z_2) = \text{arccosh}(m_p(z_1, z_2)), \quad (17)$$

where

$$m_p(z_1, z_2) = 1 + \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{2y_1 y_2}. \quad (18)$$

The connection between the hyperboloid model of the previous section and the upper half-plane model is given by relations

$$\beta = \frac{1}{y}, \quad \alpha = \frac{x}{y}, \quad \gamma = \frac{x^2 + y^2}{y}, \quad (19)$$

and

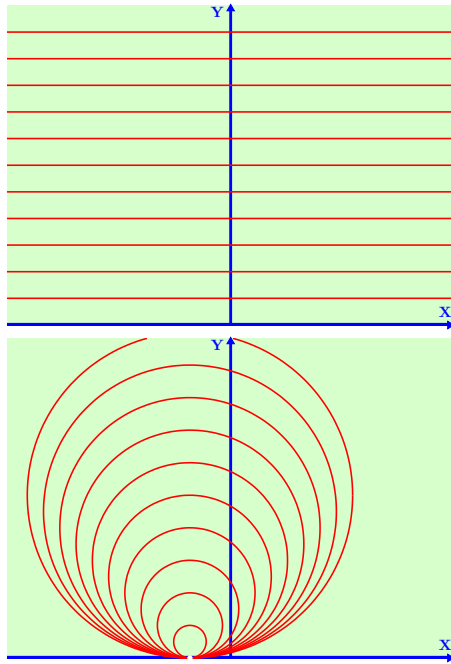


Figure 3: Flow lines of parabolic Moebius transformation.

$$x = \frac{\alpha}{\beta}, \quad y = \frac{1}{\beta}, \quad (20)$$

which provide an isometric isomorphism between them.

In the upper half-plane \mathbb{H} , the points on the imaginary axis correspond to the beams in the waist positions, and the points to the right and to the left of the imaginary axis represent converging and diverging particle beams, respectively; and the transport of the Twiss parameters in the variables z takes on the form of a Moebius map

$$z_2 = F_A(z_1) \stackrel{\text{def}}{=} -\frac{a_{22}z_1 - a_{21}}{a_{12}z_1 - a_{11}}. \quad (21)$$

As boundary of \mathbb{H} we take the set

$$\partial\mathbb{H} = \{z \in \mathbb{C} \mid y = 0\} \cup \{\infty\}, \quad (22)$$

i.e. the real axis together with the point ∞ . The action of a Moebius transformation can be extended to $\partial\mathbb{H}$ by setting

$$F_A(\infty) = -a_{22}/a_{12}, \quad F_A(a_{11}/a_{12}) = \infty, \quad (23)$$

where we use the conventions that

$$-a_{22}/0 = \infty, \quad a_{11}/0 = \infty. \quad (24)$$

Flow Diagrams of Moebius Transformations

Moebius transformations are well-known and have numerous applications in mathematics and physics, and are treated in many textbooks of complex analysis and hyperbolic geometry. Unfortunately, due to space limitation, we cannot make here a survey of even their basic properties. So, as the only example, let us discuss the geometry and classification of non-identity Moebius transformations and plot its typical phase space portraits.

Classification of non-identity Moebius transformations can be done in terms of the traces of associated with them symplectic 2×2 matrices or, equivalently, can be based

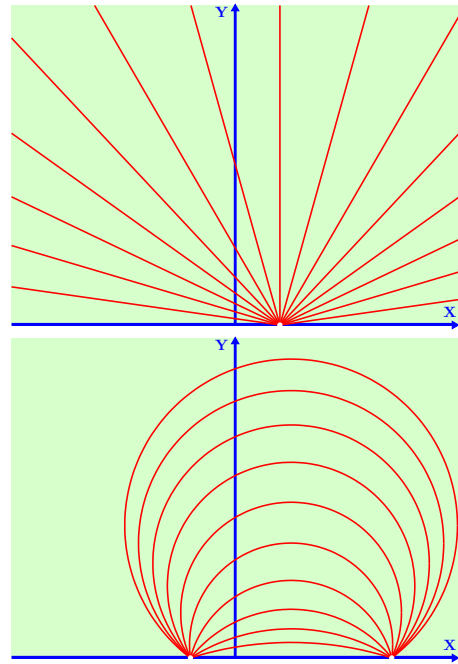


Figure 4: Flow lines of hyperbolic Moebius transformation.

on how many fixed points a given Moebius transformation (considered as acting on $\mathbb{H} = \mathbb{H} \cup \partial\mathbb{H}$) has, and whether they lie in \mathbb{H} or on the boundary $\partial\mathbb{H}$. There are three different classes of Moebius transformations:

- Elliptic: $|\text{Tr}(A)| < 2 \Leftrightarrow$ one fixed point in \mathbb{H} and none in $\partial\mathbb{H}$.
- Parabolic: $|\text{Tr}(M)| = 2 \Leftrightarrow$ one fixed point in $\partial\mathbb{H}$ and none in \mathbb{H} ; the fixed point can be finite or infinite.
- Hyperbolic: $|\text{Tr}(M)| > 2 \Leftrightarrow$ two fixed points in $\partial\mathbb{H}$ and none in \mathbb{H} ; one fixed point is always finite and the other can be finite or infinite.

Typical representatives from these different classes of Moebius transformation are shown in Figs. 2-4. Note that, looking at these figures, it is useful to have in mind that the orbits (the invariant curves, the flow lines) of a non-identity Moebius map are always intersections of either Euclidean circles or Euclidean straight lines with the upper half-plane.

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