ISBN: 978-3-95450-132-8 doi:10.18429/JACoW-IPAC2014-THPRO079

QUANTITATIVE ANALYSIS OF TRAPPING PROBABILITY FOR QUASI-INTEGRABLE TWO-DEGREE OF FREEDOM MAPS

A. Bazzani, Bologna University, Bologna, Italy
M. Giovannozzi, C. Hernalsteens*, J. Williams, CERN, Geneva, Switzerland

Abstract

A key ingredient for the Multi-Turn Extraction (MTE) at the CERN Proton Synchrotron is the beam trapping in stable islands of transverse phase space. In a previous paper a method allowing analytical estimation of the fraction of beam trapped into resonance islands as a function of the Hamiltonian parameters has been presented. Such a method applies to one-degree of freedom models of betatronic motion. In this paper, the analysis is extended to the more realistic and challenging case of two-degree of freedom systems, in which the interplay between the horizontal and vertical motion is fully included. Numerical simulations are presented and the results are discussed in detail.

INTRODUCTION

The proposal of splitting a single beam by crossing a non-linear resonance to perform a multi-turn extraction from a circular accelerator [1–3] has triggered a number of studies on the details of the splitting process. A key concept in this novel beam manipulation is adiabaticity. From a mathematical point of view the separatrix crossing always breaks the adiabatic condition and new phenomena arise (see, e.g., Refs. [4–12] for an overview).

Numerical simulations of trapping phenomena are customarily performed using simple polynomial maps. However, adiabatic theory has to be extended to be applied to maps since rigorous results are not yet available. In Ref [13] new studies are presented for the case of 2D systems described by quasi-integrable polynomial maps. The main result is that a number of scaling laws can be derived, which are robust and can be used in realistic systems. This is an important aspect for applications.

The systems considered in [13] are still too simplified for real-life applications. Therefore, in this paper we present a first attempt of extending the proposed scaling laws to 4D systems. The underlying idea is that even in a 4D system, as long as one of the two degrees of freedom is far from low-order resonances, the dynamics can be reduced to that of a 2D system, in which the additional degree of freedom can be considered as a parameter. In this picture, the extension of the results obtained for the pure 2D case is rather natural and indeed the numerical results confirm this.

THE MODEL

We consider an adiabatically modulated four-dimensional Hénon map that, on normalised co-ordinates (x, x', y, y'), describes the transverse motion of particles undergoing betatronic motion at successive cycles around a circular accelerator in the presence of a sextupole and an octupole [14]. The 4D Hénon map is given by

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix} = R(\omega_1, \omega_2) \begin{pmatrix} x \\ x' + x^2 - \chi y^2 + \kappa(x^3 - 3\chi x y^2) \\ y \\ y' - 2\chi x y - \kappa(\chi^2 y^3 - 3\chi x^2 y) \end{pmatrix} \quad (1)$$

where

$$R(\omega_1, \omega_2) = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \quad R_i = \begin{pmatrix} \cos\omega_i & -\sin\omega_i \\ \sin\omega_i & \cos\omega_i \end{pmatrix}. \quad (2)$$

and one of the frequencies is modulated, i.e., $\omega_1(n) = \omega_1(0) + \Delta\omega\, n/T$ ($T \gg 1$), with $\omega_1(0) = \pi/2 + \epsilon$, i.e., close to the fourth order resonance in the horizontal plane. The Birkhoff normal forms [14] suggest the existence of an integrable interpolating Hamiltonian for the frozen map (1)

$$H = H(\rho, \theta) = (\epsilon + h_{11}\rho_2)\,\rho + h_{20}\rho^2 + A\rho^2 \cos 4\theta, \quad (3)$$

where (ρ, θ) and (ρ_2, θ_2) are action-angle variables. The quantities h_{20}, h_{11}, A are known functions of the systems parameters, which are $\epsilon, \omega_2, \kappa, \chi$. χ represents the coupling between the two degrees of freedom, while κ provides a measure of the strength of the non-linear effects. We assume that ω_2 is far from any low-order resonance. To the low order approximation, the action-angle coordinates are related to the physical coordinates by the transformation

$$\sqrt{\rho}e^{i\theta} \simeq x - ix', \qquad \sqrt{\rho_2}e^{i\theta_2} \simeq y - iy'. \quad (4)$$

In presence of an adiabatic modulation, the interpolating Hamiltonian cannot describe the evolution of the map (1), but its action variables can be adiabatic invariants in each phase space region without low order resonances. Even if a rigorous theory for the adiabatic invariancy of 4D almost integrable symplectic maps is not yet available [15], it is plausible to assume the ansatz that the dynamics of the slowly modulated map (1) can be well described, in the considered range of parameters, by a 2D modulated Hamiltonian system that is parametrically dependent on a second integral of motion ρ_2. This means that the action ρ_2 can be considered almost constant when the orbits are not too close to a separatrix in the horizontal plane. Moreover, ρ_2 changes by a small quantity $O(\ln T/T)$ when the separatrix crossing occurs. This is confirmed by numerical simulations.

We consider $-1 \ll \epsilon < 0$, $\kappa \in [-1.9, -1.1]$, $\chi \in (0, 1]$ and $\omega_2 \in [0, 2\pi]$, so that $A < 0$, $h_{20} \geq 0$, $h_{11} \leq 0$ and

* Also at EPFL, LPAP, CH-1015 Lausanne, Switzerland.

ISBN: 978-3-95450-132-8 doi:10.18429/JACoW-IPAC2014-THPRO079

$|A| \ll h_{20}$ and Hamilton's equations read

$$\dot{\theta} = \frac{\partial H}{\partial \rho} = (\epsilon + h_{11}\rho_2) + 2\,(h_{20} + A\cos 4\theta)\,\rho \qquad (5)$$

$$\dot{\rho} = -\frac{\partial H}{\partial \theta} = 4A\rho^2 \sin(4\theta). \qquad (6)$$

Fixed points occur when $\dot{\theta} = \dot{\rho} = 0$. That is, when either

$$\theta_n = \frac{n\pi}{4}, \rho_n = \frac{-(\epsilon + h_{11}\rho_2)}{2\,[h_{20} + (-1)^n A]} \qquad (7)$$

or when

$$\rho = 0 \text{ and } \epsilon + h_{11}\rho_2 = 0. \qquad (8)$$

We have $\epsilon + h_{11}\rho_2 \le 0$ so the latter case is either impossible or we have a line of fixed points at $\rho = 0$, which is a degenerate case. The Hessian of H at (θ_n, ρ_n) is

$$\mathbb{H}|_{(\theta_n,\rho_n)} = 2(-1)^{n+1}A\rho_n^2[h_{20} + (-1)^n A] \qquad \text{i.e.} \qquad (9)$$

$$\mathbb{H}|_{(\theta_{2n},\rho_{2n})} > 0 \qquad \text{and} \qquad \mathbb{H}|_{(\theta_{2n+1},\rho_{2n+1})} < 0\,. \qquad (10)$$

and the fixed points with even n are stable and those with odd n are unstable. The oscillation frequency about the stable fixed points reads

$$\omega_e = \sqrt{\mathbb{H}} = \sqrt{-2A\rho_n^2(h_{20} + A)} = \sqrt{-\frac{A(\epsilon + h_{11}\rho_2)^2}{2(h_{20} + A)}}.$$

whereas the energy level curves containing the unstable fixed points define the separatrices. Denoting with $\rho_\pm(\theta)$ the parametrisation of the separatrix we have

$$\rho_\pm(\theta) = (\epsilon + h_{11}\rho_2)\left[\frac{-1 \mp \sqrt{\frac{2A}{A-h_{20}}}\,|\cos 2\theta|}{2\,(h_{20} + A\cos 4\theta)}\right], \qquad (11)$$

and as $|A| \ll h_{20}$ the denominator is non-zero. The separatrices divide phase space into an upper region, $\left\{(\rho,\theta) \in \mathbb{R}_{\ge 0} \times S^1 : \rho > \rho_+(\theta)\right\}$, a lower region, Σ_l, $\left\{(\rho,\theta) \in \mathbb{R}_{\ge 0} \times S^1 : \rho < \rho_-(\theta)\right\}$ and four islands, Σ_i. The area of each region is given by

$$\Sigma_i = 4\,\frac{-(\epsilon + h_{11}\rho_2)}{\sqrt{h_{20}^2 - A^2}} \tan^{-1}\left(\sqrt{\frac{-2A}{h_{20} + A}}\right) \qquad (12)$$

$$\Sigma_l = -\frac{(\epsilon + h_{11}\rho_2)}{4\sqrt{h_{20}^2 - A^2}}\left[\pi - 2\tan^{-1}\left(\sqrt{\frac{-2A}{A + h_{20}}}\right)\right]. \qquad (13)$$

As we decrease ϵ, the area below the upper separatrix, $\Sigma_i + \Sigma_l$ is increasing. For $\rho_2 = 0$, the area below the lower separatrix, Σ_l is also increasing. However, for $\rho_2 > 0$, Σ_l is non-monotonic, decreasing at first and then increasing. Figure 1 shows typical plots of $\Sigma_i + \Sigma_l$ and Σ_l in each case. This fact introduces an important difference in the evolution of the islands' surface between the 2D case, in which the behaviour for monotonic variation of the parameters was monotonic, and the 4D one, where the second degree of freedom introduces a non-monotonic variation of Σ_l.

It is worth stressing that one of the conclusions of [13] is that the existence of an interpolating Hamiltonian is essential to suggest scaling laws for the key quantities describing the trapping process even if an analytical evaluation of its parameter by using perturbation theory [14] provides results not accurate enough to be compared with numerical simulations.

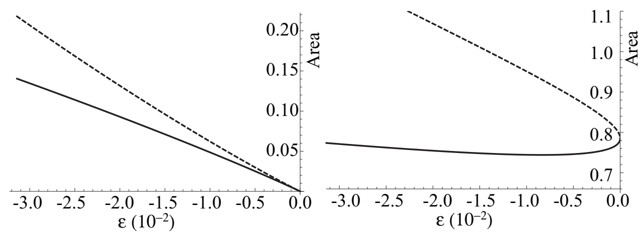

Figure 1: Variation of $\Sigma_i + \Sigma_l$ (dashed) and Σ_l (solid) for $\kappa = -1.1$, $\chi = 0.5$, $\omega_2 = 0$, $\epsilon \in [-2\pi\, 0.005, 0]$ (left: $\rho_2 = 0$, right $\rho_2 = 0.5$.

SIMULATION OF TRAPPING FRACTION

We perform numerical simulations in which a large number of particles follow the Hénon map on normalised (x, x', y, y') phase space. To first order $\sqrt{\rho}e^{i\theta} = x - ix'$ and the distributions are defined in terms of $\sqrt{\rho}$. Furthermore, a uniform distribution over a disk of radius ρ_c is assumed.

To ensure adiabaticity, the parameters should be varied over a time T much longer than the oscillation period. However, since the period of an orbit close to the separatrix is infinite, there is no finite value of T for which the motion is adiabatic over the entire phase space. Nevertheless, for sufficiently large T, the motion is adiabatic not too close to the separatrix emanating from unstable fixed points.

Figure 2 shows the simulated trapping fractions, $\mathcal{T}$, against ρ_c for $\sqrt{\rho_{2\,c}} = 0.05$ and 0.25 and for different values of the time assumed to cross the fourth-order resonance.

As for the 2D case [13], $\mathcal{T}$ features and asymptotic behaviour as a function of the typical size of the particle distribution in the horizontal degree of freedom. Also, it is clearly seen that $\mathcal{T}$ depends on ρ_2 even if the overall behaviour remains the same. Furthermore, the longer is T the larger is $\mathcal{T}$, which reflects a better adiabaticity of the trapping process. In the cases considered here the parameter χ is relatively small, i.e., the coupling between the two degrees of freedom is not too large. On the other hand, ω_2 is not too far from the third-order resonance, which seems to indicate that the behaviour observed for $\mathcal{T}$ is rather generic and similar to what obtained for the 2D case [13].

The theory predicts (see Ref. [13] and references therein) that the trapping fraction should have the form

$$\mathcal{T} \sim A + \frac{B}{\sqrt{T}} \qquad (14)$$

where the parameters A, B depend on the detail of the system under consideration. The form (14) fits the data remark-

ISBN: 978-3-95450-132-8 doi:10.18429/JACoW-IPAC2014-THPRO079

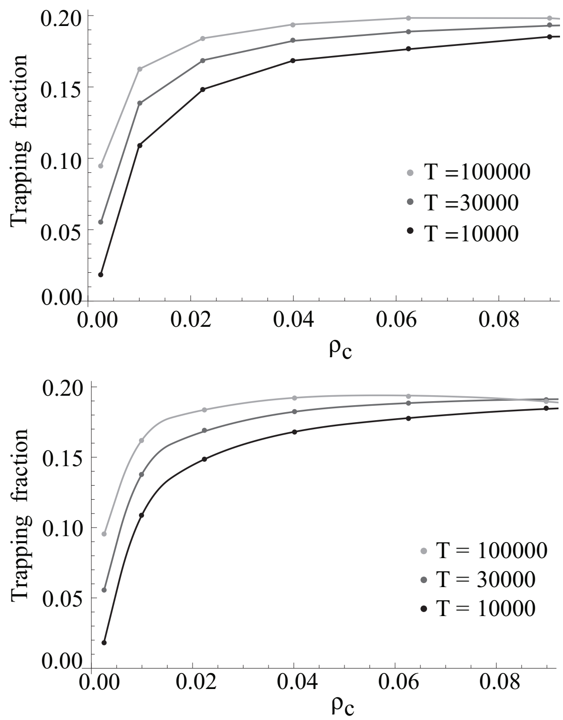

Figure 2: Simulated trapping fractions against ρ_c for $\kappa = -1.1, \chi = 0.1, \omega_2 = 0.3, T = 1, 3, 10 \times 10^4$ (upper: $\sqrt{\rho_{2\,c}} = 0.05$, lower: $\sqrt{\rho_{2\,c}} = 0.25$).

ably well in many cases, as seen in Fig. 3, where the simulation data are shown together with one such fit.

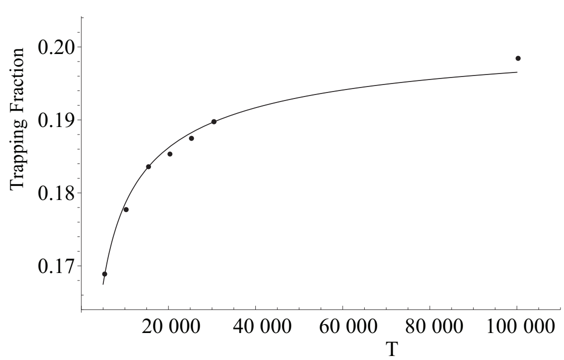

Figure 3: Simulated trapping fraction $\mathcal{T}$ against the duration, T, of the Hamiltonian evolution for $\kappa = -1.1, \chi = 0.1$, $\omega_2 = 0.3$, $\sqrt{\rho_c} = 0.25$, $\sqrt{\rho_{2\,c}} = 0.15$. The best fit of the form (14) is also shown. We find $A = 0.205$ and $B = -2.65$.

As for the 2D case [13] no trapping occurs for distribution with sufficiently small ρ_c. The value of ρ_c at which trapping begins is denoted by $R_{\min}$ and $R_{\min} \to 0$ as $T \to \infty$. Figure 4 shows a typical plot from which we estimate $R_{\min}$. The theory predicts that $R_{\min}$ and T should be related by a simple power law. Indeed, for a variation of ω_1 near a resonance $2\pi\frac{p}{q}$, $R_{\min} \propto \left(\frac{\Delta\omega_1}{T}\right)^{\frac{2p}{q}}$. This gives the linear relationship

$$\log R_{\min} = \frac{2p}{q} \log\left(\frac{1}{T}\right) + \text{constant}. \tag{15}$$

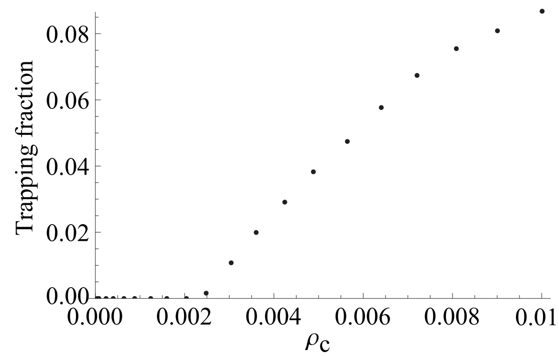

Figure 4: Simulated trapping fractions, $\mathcal{T}$, against ρ_c for $\kappa = -1.1$, $\chi = 0.1$, $\omega_2 = 0.3$, $\sqrt{\rho_{2\,c}} = 0.2$ and $T = 5000$. We estimate $R_{\min} = 0.0024$.

In the case of the fourth order resonance, $\frac{p}{q} = \frac{1}{4}$. In Fig. 5 the scaling law of $R_{\min}$ is shown in double logarithmic scale. The linear fit performed is also shown, and a remarkable

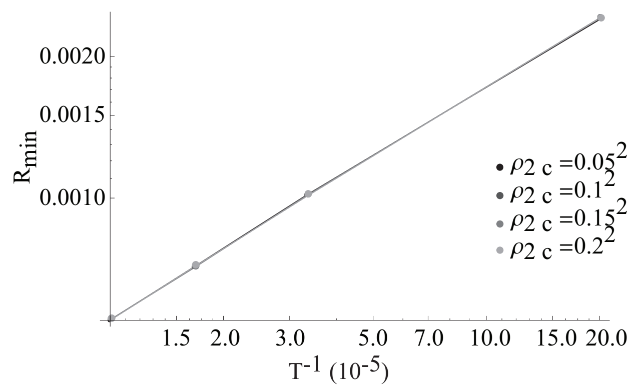

Figure 5: Power law relationship between $R_{\min}$ and T for $\kappa = -1.1$, $\chi = 0.1$, $\omega_2 = 0.3$. There is no discernible difference between curves with different values of $\rho_{2\,c}$.

agreement is visible, with a slope of 0.49, in agreement with theory. It turns out that $\rho_{2\,c}$ has very little effect on $R_{\min}$.

CONCLUSION

In this paper, a first analysis of scaling laws for trapping due to crossing of a non-linear resonance for 4D quasi-integrable maps has been presented and discussed.

These results follow a detailed characterisation of the 2D case [13]. The numerical results seem to indicate that the same scaling laws are describing the process in 2D and 4D, at least whenever the dynamics of the second degree of freedom is not affected by low-order resonances. Such an assumption is certainly very reasonable in view of applications in circular particle accelerators.

The next step will consist of extending the range of values used for the parameters considered for the 4D Hénon map and to probe the impact of correction of the non-linear coupling between the two degrees of freedom. Such a parameter is proportional to $h_{1,1}$ and has been probed also in a number of experimental tests in the CERN PS machine [16], thus providing an interesting benchmark between theory and simulations and real beam dynamics.

ISBN: 978-3-95450-132-8 doi:10.18429/JACoW-IPAC2014-THPRO079

REFERENCES

[1] R. Cappi, M. Giovannozzi, Phys. Rev. Lett. 88, 104801 (2002).

[2] R. Cappi, M. Giovannozzi, Phys. Rev. ST Accel. Beams 7, 024001 (2004).

[3] A. Franchi, S. Gilardoni, and M. Giovannozzi, Phys. Rev. ST Accel. Beams 12, 014001 (2009).

[4] A. Neishtadt, Sov. Phys. Dokl., 20, 189, 1975.

[5] A. Neishtadt, J. Appl. Math. Mech., 39, 594, 1976.

[6] A. Neishtadt, Sov. J. Plasma Phys., 12, 568, 1986.

[7] J. L. Tennyson, J. R. Cary, and D. F. Escande, Phys. Rev. Lett. 56, 2117, 1986.

[8] J. R. Cary, D. F. Escande, and J. L. Tennyson, Phys. Rev. A 34, 4256, 1986.

[9] Y. Elskens, D. F. Escande, Nonlinearity 4, 615, 1991.

[10] J. Henrard, Dynamics Reported 2, 117, 1993.

[11] Y. Elskens, D. F. Escande, Physica D 62, 66, 1993.

[12] B. V. Chirikov and V. V. Vecheslavov, Journal of Experimental and Theoretical Physics, 90-3, 562, 2000.

[13] A. Bazzani, C. Frye, M. Giovannozzi, C. Hernalsteens, Phys. Rev. E 89, 042915 (2014).

[14] A. Bazzani G. Servizi, E. Todesco, G. Turchetti, CERN Yellow Report 94-02, 1994.

[15] A. I. Neishtadt, A. A. Vasiliev, A. V. Artemyev, Regular and Chaotic Dynamics 18, p. 686 (2013).

[16] M. Giovannozzi et al., "First Results for the Beam Commissioning of the CERN Multi-Turn Extraction", IPAC'10, Kyoto, May 2010, THOBMH02, (2010), http://www.JACoW.org