BUNCH DYNAMICS THROUGH ACCELERATOR COLUMN*

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Abstract

The differential equations for the bunched beam envelope through an axially symmetric DC accelerator are derived. An example of such a case is the electron gun with rf-modulated triode. In the case of no space charge, a particle's total energy is conserved, so the longitudinal evolution is simple: particles of same energy are a fixed time increment apart and this implies in first order that their separation is proportional to their speed. However, with space charge, the longitudinal force depends upon the bunch length, so we need equations that track this parameter.

INTRODUCTION

As alternative to multi-particle simulations, envelope calculations are computationally much faster. However, existing envelope codes were either DC, non-relativistic, or non-space-charge.

The mathematical fomalism for this technique, including space charge, was established by Frank Sacherer[1]. The space charge forces depend crucially upon the bunch dimensions in configuration space, so it is important that these be tracked. In other words, it is insufficient to use a formalism that first integrates the equations of motion to derive the transfer matrices, and apply space charge effects afterwards. Some implementations such as `TRANSPORT` and `TRACE3D` divide standard elements into (hopefully sufficiently short) segments interleaved with space charge "lenses". This is crude, approximate and non-adaptive.

`TRANSOPTR`[2] uses the envelope formalism, but did not until now include the case of beam high intensity bunches being accelerated with a DC potential difference accelerator. This case is of particular importance for modeling the grid-modulated elecron gun, as the bunch is created very short (typically <1 mm) then grows rapidly as it is accelerated from rest, finally attaining a length of a few cm (see Fig. 1).

THEORY

Hamiltonian

With the distance along the reference trajectory s as the independent variable, the Hamiltonian for the case with axial electric potential, no magnetic fields, and initially ignoring space charge, is

$$H(x,P_x,y,P_y,t,E;s) = \\ = -\sqrt{\left(\frac{E-q\Phi}{c}\right)^2 - m^2c^2 - P_x^2 - P_y^2} \tag{1}$$

The potential is given by

$$\Phi(r,s) = \phi(s) - \frac{1}{4}\phi''(s)r^2 + O(r^4) \tag{2}$$

This is time-independent, so E is conserved. Further, particles launched with identical coordinates except separated in time by Δt will remain separated by this time, but of course this means their spatial separation increases as long as they are accelerating.

The 4 parameters (x,P_x,y,P_y) are small so we expand to second order:

$$H \approx -P - \frac{q\phi''}{4\beta c}(x^2+y^2) + \frac{P_x^2+P_y^2}{2P} \tag{3}$$

Here P is an abbreviation containing E:

$$P = \sqrt{\left(\frac{E-q\phi}{c}\right)^2 - m^2c^2} \tag{4}$$

Expanding about the reference energy $E_0 = E - \Delta E$ with $\Delta E \ll E_0$,

$$P = P_0 + \frac{\Delta E}{\beta c} - \frac{(\Delta E)^2}{2\beta^3\gamma^3 mc^3}, \tag{5}$$

$(P_0 = mc\sqrt{\left(\frac{E_0-q\phi}{mc^2}\right)^2 - 1} = mc\sqrt{\gamma^2-1} = mc\beta\gamma.)$

$$\begin{aligned} H &= -P_0 - \frac{\Delta E}{\beta c} + \\ &+ \frac{(\Delta E)^2}{2\beta^3\gamma^3 mc^3} - \frac{q\phi''}{4\beta c}(x^2+y^2) + \frac{P_x^2+P_y^2}{2P} \end{aligned} \tag{6}$$

Note that through ϕ, P_0 is dependent on s.

At this point the canonical variables are still t and $-E$. To transform to $(\Delta t, -\Delta E)$, consider that

$$\frac{ds}{dt} = \beta c = \frac{Pc^2}{E-q\Phi} \tag{7}$$

so the reference particle's time coordinate is $t_0 = \int \frac{ds}{\beta_0(s)c}$. This suggests a generating function

$$G = -\left(t - \int \frac{ds}{\beta_0 c}\right)(\Delta E + E_0) \tag{8}$$

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(Check: $\frac{\partial G}{\partial t} = -E$, $\frac{\partial G}{\partial(-\Delta E)} = \Delta t$.) The Hamiltonian gets the added term $\frac{\partial G}{\partial s} = \frac{\Delta E + E_0}{\beta_0 c}$:

$$\begin{aligned} H_{\Delta t} &= \left(\frac{E_0}{\beta c} - P_0\right) + \\ &+ \frac{(\Delta E)^2}{2\beta^3\gamma^3 mc^3} - \frac{q\phi''}{4\beta c}(x^2+y^2) + \frac{P_x^2+P_y^2}{2P} \end{aligned} \tag{9}$$

(To keep the notation uncluttered, we drop the 0 subscripts on the β's and γ's; it is understood that they refer to the relativistic parameters of the reference particle.) The first term is ignorable for our purposes as it depends upon s only and not on any of the canonical variables.

Finally, we wish to transform from $(\Delta t, -\Delta E)$ to $(\Delta z, \Delta P) = (-\beta c\Delta t, \Delta E/(\beta c))$. (The reason for the sign change is as follows: an early arrival implies $\Delta t < 0$, but this means the particle is **ahead** so $\Delta z > 0$.) The generating function is

$$G = -\beta c \Delta t \Delta P \tag{10}$$

(Check: $\frac{\partial G}{\partial \Delta t} = -\Delta E$, $\frac{\partial G}{\partial(\Delta P)} = \Delta z$.) The term to be added to the Hamiltonian is $\frac{\partial G}{\partial s} = \frac{\beta'}{\beta}\Delta z\Delta P$:

$$H_{\Delta z} = \frac{\beta'}{\beta}\Delta z\Delta P + \frac{(\Delta P)^2}{2\gamma^2 P} - \frac{q\phi''}{4\beta c}(x^2+y^2) + \frac{P_x^2+P_y^2}{2P} \tag{11}$$

$\beta' = \frac{d\beta}{ds}$ can be found from ϕ using $\gamma mc^2 = E_0 - q\phi$ and E_0=constant: $\gamma' = -\frac{q\phi'}{mc^2}$, $\beta' = \frac{\gamma'}{\beta\gamma^3}$.

Infinitesimal Transfer Matrix

A convenient and useful way of representing the equations of motion through the optical element is the so-called infinitesimal transfer matrix approach[1]. The infinitesimal transfer matrix $F(s)$ is defined as $(T-I)/ds$ where T is the transfer matrix from s to $s+ds$ and I is the identity matrix. With this definition one has for individual particles

$$\frac{d\mathbf{x}}{ds} = F\mathbf{x}, \qquad \text{where} \qquad \mathbf{x} \equiv \begin{pmatrix} x \\ P_x \\ y \\ P_y \\ \Delta z \\ \Delta P \end{pmatrix}. \tag{12}$$

Beams of particles are conveniently represented by the so-called σ-matrix; the elements of which represent second order moments of the beam[1]. The σ-matrix and the transfer matrix M transform through the system according to the equations

$$\frac{d\sigma}{ds} = F\sigma - \sigma F^T, \tag{13}$$

$$\frac{dM}{ds} = FM. \tag{14}$$

where F^T is the transpose of F.

Now that the Hamiltonian for linear motion (eqn. 11) has been obtained, it is a simple matter to find the infinitesimal transfer matrix F. Writing the equations of motion ($x' = \partial H/\partial P_x$, $P_x' = -\partial H/\partial x$, etc.) in the form of Eqn. 12, the following F-matrix is found for the axially symmetric DC accelerator:

$$F = \begin{pmatrix} 0 & \frac{1}{P} & 0 & 0 & 0 & 0 \\ \frac{q\phi''}{2\beta c} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{P} & 0 & 0 \\ 0 & 0 & \frac{q\phi''}{2\beta c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\beta'}{\beta} & \frac{1}{\gamma^2 P} \\ 0 & 0 & 0 & 0 & 0 & -\frac{\beta'}{\beta} \end{pmatrix}. \tag{15}$$

SPACE CHARGE

Space charge forces depend recursively upon the σ-matrix elements, and are simply added to the focusing elements $F_{2n,2m-1}|_{n,m=1,2,3}$ of the element's infinitesimal transfer matrix such as eqn. 15 above.

$$F = \begin{pmatrix} 0 & \frac{1}{P} & 0 & 0 & 0 & 0 \\ K_x + \frac{q\phi''}{2\beta c} & 0 & K_{xy} & 0 & K_{xz} & 0 \\ 0 & 0 & 0 & \frac{1}{P} & 0 & 0 \\ K_{xy} & 0 & K_y + \frac{q\phi''}{2\beta c} & 0 & K_{yz} & 0 \\ 0 & 0 & 0 & 0 & \frac{\beta'}{\beta} & \frac{1}{\gamma^2 P} \\ K_{xz} & 0 & K_{yz} & 0 & K_z & -\frac{\beta'}{\beta} \end{pmatrix}. \tag{16}$$

In the symmetric case with axes along x, y, and z, the cross terms disappear; $K_{xy} = K_{xz} = K_{yz} = 0$. (In the general case, we make the calculations in the frame of the ellipsoid's axes, and then perform a rotation to the lab frame.)

This technique is used in the code TRANSOPTR, as described by de Jong[2]. The resulting equations can only be solved numerically.

The given references[1, 2] treat space charge in the non-relativistic regime, where (for particle charge q, bunch charge Q)

$$K_x = \frac{qQ}{4\pi\epsilon_0}\frac{1}{\beta c}R_D(\sigma_{33}, \sigma_{55}, \sigma_{11}) \tag{17}$$

$$K_y = \frac{qQ}{4\pi\epsilon_0}\frac{1}{\beta c}R_D(\sigma_{55}, \sigma_{11}, \sigma_{33}) \tag{18}$$

$$K_z = \frac{qQ}{4\pi\epsilon_0}\frac{1}{\beta c}R_D(\sigma_{11}, \sigma_{33}, \sigma_{55}) \tag{19}$$

where R_D is the Carlson elliptic integral

$$R_D(u,v,w) = \frac{3}{2}\int_0^\infty \frac{dt}{(t+w)\sqrt{(t+u)(t+v)(t+w)}}$$

and σ_{ij} is the usual TRANSPORT-notation σ-matrix, but using $\sqrt{5}$ times rms values for the case of the non-uniformly-populated ellipsoid, e.g. $\sigma_{11} = \sqrt{5\langle x^2\rangle}$.

It was not obvious that TRANSOPTR was correct in the relativistic regime. There are two effects that need to be considered to generalize the equations: the space charge magnetic field, and bunch length contraction. For detailed derivations, the interested reader is referred to the Ph.D. thesis of Fubiani[3]. The first effect requires dividing the space charge force by γ^2. The second requires that the Carlson elliptic integrals' arguments be modified. The result is:

$$K_x = \frac{qQ}{4\pi\epsilon_0}\frac{1}{\beta\gamma c}R_D(\sigma_{33},\gamma^2\sigma_{55},\sigma_{11}) \quad (20)$$

$$K_y = \frac{qQ}{4\pi\epsilon_0}\frac{1}{\beta\gamma c}R_D(\gamma^2\sigma_{55},\sigma_{11},\sigma_{33}) \quad (21)$$

$$K_z = \frac{qQ}{4\pi\epsilon_0}\frac{1}{\beta\gamma c}R_D(\sigma_{11},\sigma_{33},\gamma^2\sigma_{55}) \quad (22)$$

These together with the evolution equation for the σ-matrix (13) and the accelerator column infinitesimal ransfer matrix (16) form a closed system of equations. An example calculation is shown in Fig. 1.

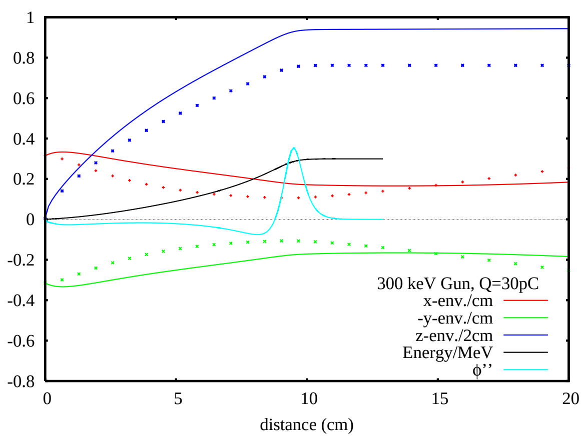

Figure 1: Beam envelopes calculated from a cathode, $Q = 30$ pC bunches accelerated to 300 keV in a distance of 9.5 cm. The case without space charge is shown dotted for comparison.

Long-bunch Limit

An interesting limit is the long bunch, since this can be approximated as a continuous beam with current I. First of all, it is clear that for this limit to apply, bunch length $\gg$ transverse size is not a necessary condition. Rather, γ times bunch length $\gg$ transverse size. This means that for example a 1 mm long by 1 mm wide electron bunch is already well into the long-bunch regime with energy of 10 MeV.

Secondly, in the long bunch regime, the Carlson integrals governing transverse space charge are $\propto (\gamma\sqrt{\sigma_{55}})^{-1}$, or the inverse bunch length augmented by the factor γ. Specifically, for $u \gg (v, w)$,

$$R_D(u,v,w) \to \frac{3}{\sqrt{uw}(\sqrt{v}+\sqrt{w})} \quad (23)$$

Substituting 23 into the equations for K:

$$K_x = \frac{qI}{4\pi\epsilon_0}\frac{1}{\beta^2\gamma^2c^2}\frac{1}{a(a+b)} \quad (24)$$

$$K_y = \frac{qI}{4\pi\epsilon_0}\frac{1}{\beta^2\gamma^2c^2}\frac{1}{b(a+b)} \quad (25)$$

where we use an appropriately-defined (distribution dependent) effective current $I \propto \frac{Q}{\sqrt{\sigma_{55}}\beta c}$ and $a, b \propto \sqrt{\sigma_{11}}, \sqrt{\sigma_{33}}$, so these can be seen to reduce to the Kapchinsky-Vladimirsky equations.

REFERENCES

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