# AN ANALYTICAL LAGRANGIAN MODEL FOR ANALYZING TEMPERATURE EFFECTS IN INTENSE NON-NEUTRAL BEAMS*

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*Abstract*

We construct a Lagrangian warm-fluid model for describe the behavior of a inhomogeneous charged-particle beam, under the effects of a constant solenoidal focusing field. The equations of motion are derived for an adiabatic process, with a state equation originated from the ideal gas law. In the end, the model is compared with self-consistent simulation and is used to explain emittance growth and jets of particle even when the system is out of equilibrium.

## INTRODUCTION

High-intensity charged-particle beams are used in several areas of Physics. We can mention as an illustration, high-energy colliders, particle accelerators and vacuum electron devices [5]. In all cases quoted, the beam lose particles in the acceleration process due the inumerous sources of non-linear effects. These ejected particles can collide with the walls chambers of the acceleration device, causing damage and reduction of the beam energy [6].

Gluckstern [1] show that initial envelope oscillations of mismatched homogeneous beams, induce formation of large scale resonant islands beyond the beam border. Beam particles are captured by these resonant islands resulting in emittance growth and relaxation. However, a more closed questions with the aforementioned problems involve the study of the beam before the halo be formed [2], and also, the relevant phenomena created.

One of these phenomena is the wave breaking, which seems to be studied for homogeneous and cold beams, firstly by Dawson[7]. This phenomena is associated with the jets of particles, and its control is relevant to avoid the lost of energy in accelerators. For the beams that are inhomogeneous, and still cold, the process is similar to the studies of Dawson, however is necessary a controlling role for adjust envelope size mismatches and degrees of inhomogeneities [3]. Unfortunately, when the beam is not more cold, we can't use wave breaking to monitor the jets of particles, owing to the particles spread which are already present at initial times.

In view of the whole problem, we developed a Lagrangian fluid model, taking into account temperature effects by means an isotropic radial pressure profile. We focused our analysis in the first jets of particles, considering that these particles will be the first to form the halo. For detect the jets, we start investigating the dynamics from rms emittance [4] point of view, splitting it in thermal and fluid part. And then, we finish the paper comparing the Lagrangian model with self-consistent simulations.

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## ANALYTIC LAGRANGIAN MODEL

Our aim in this section is modeling warm but space charge dominated beams, considering the macroscopic fluid description when the beam has cylindrical symmetry. If we sit in a fluid element and move with it as the fluid moves in the defined $z$ direction (using Lagrangian fluid description), is possible to associate a center of mass coordinate $r$ and a center of mass velocity $\dot{r}$, which represents the mean velocity of all individual particles which make up the fluid at $r$. Now, if we wander with this "beam fluid element" through a solenoidal channel (governed by linear forces, in the paraxial approximation [6]) and supposing that all the particles have the same charge (with a repulsive force given by gauss law), we can estimate the Lagrangian of this beam fluid element by integrating a distribution function, $f(\mathbf{r},\mathbf{v})$, for an arbitrary element of the phase space area, $dA$, viz:

$$L(r,\dot{r}) = \int_A \Big(\frac{\dot{r}^2}{2} - \frac{r^2}{2} + Q(r_0)Log(r)\Big) f(\mathbf{r},\mathbf{v})dA \quad (1)$$

where, $r_0$ represents the initial fluid coordinate at z=0, $Q(r_0) = 2\pi\int n_0 r_0 dr_0$ is the measure of the charge between 0 and $r_0$, $n_0$ is initial parabolic inhomogeneous density profile, defined by $n_0 = 1/\pi r_b^2(1+\chi(2r_0^2/r_b^2-1))$, $r_b$ is the envelope radius, $\chi$ is the inhomogeneous amplitude factor. Considering that the center of mass velocity $\dot{r}$ can be described as an average velocity ($\bar{\dot{r}}$) plus a dispersion ($\delta\dot{r}$) as follow, $\dot{r} = \bar{\dot{r}} + \delta\dot{r}$, we can redefine our Lagrangian as average quantity:

$$L(r,\dot{r}) = \frac{\dot{r}^2}{2} - \frac{r^2}{2} + Q(r_0)Log(r) + P(r,r_0) \quad (2)$$

where the new term $P(r,r_0)$ depict the isotropic transversal pressure provided by the relationship, $P(r,r_0) \equiv \int(\delta r')^2 f(\delta r')d^2\delta r'$[6]. Now for determine the pressure profile, we will suppose an adiabatic process, where the equation of state can be written as: $P/n^\gamma = constant$; where $\gamma = 2$ represent a two-dimensional system. Since we know that the density $n$ can be defined through continuity equation, $n = (n_0 r_0)/r(\partial r/\partial r_0)$, and $n_0$ is linked with the initial temperature by the ideal gas law, $P_0(r_0) = n_0 T_0$, the transversal pressure takes the form: $P(r,r_0) =$

$(n_0 r_0^2 T_0)/(r^2(\partial r/\partial r_0)^2)$; where $T_0$ is the constant initial temperature and $\partial r/\partial r_0$ is the compressibility factor.

Applying the Euler-Lagrange equation to the Lagrangian $L(r,\dot{r})$ and defining the compressibility factor as $C = \partial r/\partial r_0$ and its derivative with respect to $r_0$ as $D = \partial r^2/\partial r_0^2$, we have:

$$\ddot{r} = -r + \frac{[r_0^2\chi - r_b^2(\chi-1)]r_0^2}{r_b^4 r} + \frac{2r_0^2 T_0[2r_0^2\chi - r_b^2(\chi-1)](C^2 - rD)}{\pi r_b^4 C^4 r^3} \quad (3)$$

Now, to form a closed set, we need more two ordinary differential equations (one for $C''$ and other for $D''$). We can get them by deriving the equation (3) twice, with respect to $r_0$, and truncating the result until second order derivative. In the end, we will have a set of self-consistent differential equations formed by $\ddot{r}$, $\ddot{C}$ and $\ddot{D}$.

## THERMAL AND DIRECTIVE EMITTANCE

When the system has a considerable number of nonlinear forces, it is convenient use the concept of rms emittance defined by Sacherer and Lapostolle [4]. This proposal allows us to write the transversal rms emittance in function of the seconds statistical moments of the particles

$$\varepsilon_{rms}^2 = 4(\langle r^2\rangle\langle \mathrm{v}^2\rangle - \langle \mathbf{r}\cdot\mathbf{v}\rangle^2) \quad (4)$$

Without losing generalization, we can rewrite the total velocity $\mathbf{v}$ for a group of particles as a local average plus a dispersion of it's mean, $\mathbf{v} = \overline{\mathbf{v}} + \delta\mathbf{v}$, where $\mathbf{v}$ is the total velocity, which can be defined in terms of its components as, $\mathbf{v} = \dot{r}\hat{\mathbf{e}}_r + (L/r)\hat{\mathbf{e}}_\theta$. Then, the rms emittance will be written as:

$$\varepsilon_{rms}^2 = 4[\langle r^2\rangle\langle(\overline{\mathbf{v}}+\delta\mathbf{v})^2\rangle - \langle \mathbf{r}\cdot(\overline{\mathbf{v}}+\delta\mathbf{v})\rangle^2] \quad (5)$$

Knowing that $\langle\delta\mathbf{v}\rangle = \overline{\mathbf{v}} - \overline{\overline{\mathbf{v}}} = \mathbf{0}$, thus we have;

$$\varepsilon_{rms}^2 = 4[\langle r^2\rangle\langle\overline{\mathrm{v}}^2\rangle + \langle r^2\rangle\langle\delta\mathrm{v}^2\rangle - \langle\mathbf{r}\cdot\overline{\mathbf{v}}\rangle^2] \quad (6)$$

Now, we can identify terms dependent of $\overline{\mathrm{v}}$ and $\delta\mathrm{v}$, which lead us to separate these two dependences in two new quantities, defined as thermal emittance and directive emittance:

$$\varepsilon_{dir}^2 = 4[\langle r^2\rangle\langle\overline{\mathrm{v}}^2\rangle - \langle\mathbf{r}\cdot\overline{\mathbf{v}}\rangle^2] \quad (7)$$
$$\varepsilon_{the}^2 = 4[\langle r^2\rangle\langle\delta\mathrm{v}^2\rangle] \quad (8)$$

So, rewriting the rms emittance in a compact form:

$$\varepsilon_r^2 = \varepsilon_{dir}^2 + \varepsilon_{the}^2 \quad (9)$$

By splitting the rms emittance in two parts, we can benefit by looking separately the effects of average oscillation (directive emittance) and velocity dispersion (thermal emittance). However, for the studies with particle ejection and halo formation is more convenient just work with thermal emmitance, because of its proportionality with $\delta\mathrm{v}$.

## SELF-CONSISTENT SIMULATIONS

The computer simulations were carried out considering full azimuthal symmetry, where one can use Gauss law in order to write the governing equation for any particle in the beam [3]:

$$\ddot{r}_i = -\kappa r_i + \frac{Q(r_i)}{r_i} + \frac{L_i^2}{r_i^3}; \quad (10)$$

where $i$ represent the index of each particle in the Larmor frame. The first term on the right-hand side of equation (10) depict the focusing external magnetic field; the second term is the electric field due the repulsion of the particles. Since the beam has and initial temperature, the third term will represent the angular moment. The primes indicates the derivative with respect to the longitudinal z coordinate. The focusing factor is $\kappa \sim B^2$, where $B$ is the axial, constant, focusing magnetic field and $Q(r_i)$ is the measure of beam charge (in fact, is the perveance) up to the position $r_i$.

For be consistent with the Lagrangian analytical model, in equation (3), we will use the same density profile in the initialization as before, that is, a parabolic shape defined by: $n_0(r_i) = \frac{N_p}{\pi {r_b}^2}\left[\chi\left(\frac{2{r_0}^2}{{r_b}^2} - 1\right) + 1\right]$; where $N_p$ is the total number of beam particles per unit length, the parameter $-1 \le \chi \le 1$ controls the non-homogeneity charge distribution, $r_0$ is the initial position, and $r_b$ is the initial beam size. In the same way, the initial velocity (in the numerical simulations) can be related with the initial temperature (in the analytical model) according with a water-bag distribution function. For this, we suppose $f = \Theta(v_0 - |v|)/2v_0$, defined in the interval $-v_0 < v < v_0$, where $\Theta(v_0 - |v|)$ is the Heaviside function; so, the initial temperature $T_0$ can be estimated by its proportionality to the mean square-velocity, $\langle v^2\rangle$. Then, if we integrate the velocity over all the initial conditions, we have the relationship: $T_0 \propto \langle v_0^2\rangle = \int_{-v_0}^{v_0} v^2 f dv = v_0^2/3$.

## LAGRANGIAN MODEL VERSUS SELF-CONSISTENT SIMULATIONS FOR A PARTICLE EJECTION EXPLANATION

Now we can use the thermal emmitance concepts to distinguish the behavior between two beams initialized with non-zero temperature, how it is shown in figure 1.

Among these two thermal emittance curves, one of them was created with a very small initial temperature (curve (a)) and other the one, with a five hundred higher value (curve (b)). As we can see, the curve (a) reveals that until $z \approx 119$ (time adimensional unit), the beam oscillates around zero value, which means that all the particles are oscillating around their equilibrium point without changing the initial beam area. After $z \approx 119$, the beam undergo an immediately rising of thermal emittance, indicating that the first jets of particles are going to the halo. These jets are followed by others, making the thermal emittance grows until the beam reach the relaxation, with happen around $z \approx 175$. This behavior is also identified in cold beams

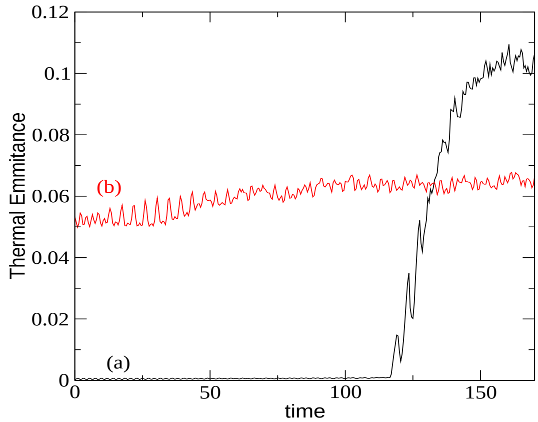


Figure 1: Thermal emittance curves for two different initial velocities. The black line is a beam initialized with $v_0 = 0.0001$ and the red line represent a beam with initial velocity $v_0 = 0.05$. In both cases the beams are matched, with inhomogeneity $\chi = 0.6$, in the initial state.

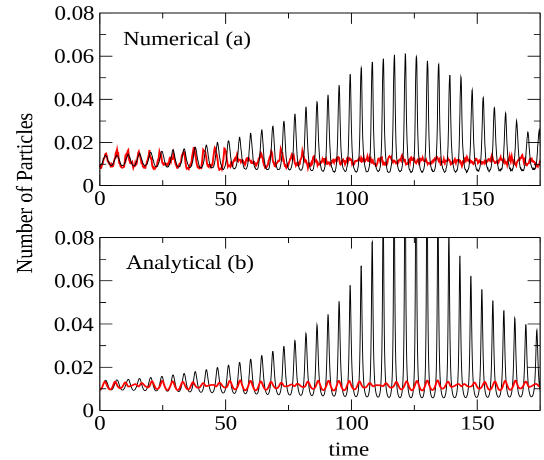


Figure 2: Local Normalized number of particles. The black line defines the beam with initial velocity $v_0 = 0.0001$ and the red line, with 0.05. In (a), we have self-consistent numerical simulation of 10.000 particles and in (b), the analytical Lagrangian model. For both case we start the system in $r_0 = 0.4$, with $\chi = 0.6$ and initial envelope mismatched $r_b = 1.0$.

[3], what suggests that the cold beam approximation is still valid for very low temperature ranges.

More remarkable differences can be noted in curve (b), when the initial temperature is higher than (a). Since now we have a low, but thousand times higher, initial temperature ($T_0 = 0.00083333$), the particles have an initial dispersion in velocity space. So, as the beam interact with the external magnetic field and the space charge effects, the particles are being detached from the beam core ($z = 0$ to $z = 32$), however, without sufficient velocity to form an halo, what keep them oscillating just near the beam core. Only after $z = 32$ that the beam eject the first particles with considerable energy. This launch procedure is carried out gradually and slowly, in opposition of curve (a), in which a big amount of particles is ejected by means of intense jets.

Outwardly, the explanation of this two distinct behavior, comes from the cumulation of particles in certain regions of the phase space. As the beam act as a non-linear oscillator, in which each particle oscillates around their equilibrium point; for a given time, they can group together in some regions of phase space. This grouping process seems to be more intense for low temperatures, than higher temperature, as we can verify in figure 2.

For this illustrative example, first, we choose a region in the beam where the first jets occurs (we could designate other one, which the behavior will be similar) and after, we monitor this region counting the normalized number of particles. This procedure is depicted in the panel (a) and (b), where we compare two equivalent beams displayed in the figure 1, for numerical and analytical solutions respectively. At a glance, we can see that the black lines, that represent the beams with low temperature, have a modulated growing in the amount of particle, reaching its maximum near $z \approx 119$, that non-coincidentally, is the point that we have the first jets of particles, according with the figure 1.

On the other hand, the red line keeps a small oscillation, don't changing too much the amount of particles. The foregoing indications make us can claim that, the temperature effects reduce the cumulation of particles in punctual areas of the beam, spreading them and making the jets weaker.

## FINAL REMARKS

In this work, we employed Lagrangian techniques to explain the halo formation for inhomogeneous beams with temperature. The equations of motion were derived for an adiabatic process, and the estimates compared with self-consistent simulations.

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