

# NONLINEAR DYNAMICS INDUCED BY 1-D MODEL OF PINCHED ELECTRON CLOUD

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## Abstract

The presence of an electron cloud in an accelerator generates a number of interesting phenomena. In addition to electron-driven beam instabilities, the electron “pinch” occurring during a beam-bunch passage gives rise to a highly nonlinear force experienced by individual beam particles. A simple 1-dimensional model for the effect of the electron pinch on the beam reveals a surprisingly rich dynamics. We present the model and discuss simulation results.

## INTRODUCTION

The interaction of the electron cloud in a synchrotron with the passing beam creates a rich jungle of phenomena. The electron-driven instabilities have received full attention as the time scale of their appearance is fast. It has, however, been recently recognized that even the stable dynamics of the coupled system bunched beam / electron cloud, can originate a complex beam dynamics leading to phenomena of periodic resonance crossing [1]. The time scale of these phenomena is much longer. Reference [2] has studied in detail the “pinch” dynamics of the electrons during the passage of a bunched proton beam. Under the action of the proton Coulomb field, the electrons undergo a pinch which creates a complex structure of “rings”. This structure weakly feeds back onto the tunes of the proton beam according to the actual location of these particles in the bunch frame. The global effect is an induced periodic tune modulation leading to possible resonance crossing. As shown in Ref. [2], this phenomenon closely resembles that encountered in the case of a stored high intensity bunched beam, where the periodic tune modulation is due to space charge [3]. The challenge in predicting the evolution of the proton bunch lies in the slowness of the full phenomena. The periodic crossing of a resonance creates a very slow beam diffusion which can be detected only on a long time scale (hundreds of synchrotron oscillations). This diffusion-generating mechanism is highly relevant for a collider like the LHC, but also in the case of space-charge driven periodic resonance crossing, as found for SIS100 [4]. The understanding of this coupled beam dynamics is essential for very long term prediction. A suitable approach uses a frozen model. Here the effect of the pinch is modeled with a simplified EC structure, where the structure of the pinch is kept the same from turn to turn while allowing the proton bunch dynamics to evolve. The main approximation made in Ref. [2] was to consider the EC pinch in an axis-symmetric approximation. Although the approximation allowed an estimate for accelerators like LHC, RHIC, and SIS100 (for the latter with space charge), a discussion of the underlying nonlinear dynamics was left unexplored. In this paper we extend a simplified model, earlier presented

in Ref. [5], and discuss its dynamical properties, gaining some insight into the complex dynamics governing a “resonance scattering” regime induced by the electron cloud.

## THE ELECTRON CLOUD STRIPE MAP

We here consider the beam dynamics induced by the pinched electron cloud for protons moving only in one plane, so that the system becomes 1-dimensional. We also assume that the field created by the pinched electrons can, locally in  $z$ , be approximated by the one of an ideal planar structure consisting of 2 infinitely wide parallel planes of negligible thickness, and located at a vertical distance  $y_{EC} = \pm R_1$  from the axis. The Coulomb field produced by this system is  $E = 0$  between the two planes, and  $E = \pm \text{const.}$  outside them (Fig. 1). The quantity  $R_1$  mod-

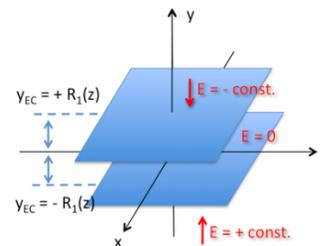


Figure 1: Schematic of the 1-dimensional model of the electron cloud pinch.

els the extent of the pinch: at the beginning of the bunch  $R_1 = 0$  and later along the bunch it grows according to the beam and EC characteristics as discussed in Ref. [2], typically  $R_1 = R'_1(z - z_p)$ , where  $z_p$  is the bunch longitudinal location at which the pinch starts. Consider a proton located at the head of a bunch ( $z < 0$ ) near  $z_p$ . This proton will experience the force from the EC plane located at  $y = 0$ . The one turn map expressed in Courant-Snyder (CS) coordinates of the system pinched EC + accelerator is then

$$\begin{pmatrix} \hat{y}_1 \\ \hat{y}'_1 \end{pmatrix} = \begin{pmatrix} \cos \omega & \sin \omega \\ -\sin \omega & \cos \omega \end{pmatrix} \begin{pmatrix} \hat{y}_0 \\ \hat{y}'_0 + \mathcal{F}(\hat{y}_0) \end{pmatrix} \quad (1)$$

with  $\omega = 2\pi Q_y$  the phase advance, and  $\mathcal{F}(\hat{y}_0) = \text{sign}(\hat{y}_0)\hat{F}$  representing the Coulomb field generated by the plane of electrons located at  $\hat{y} = 0$ . The function “sign” incorporates the discontinuity of the Coulomb electric field generated by the electrons. With the change of variable  $\mathbf{y} = (\hat{y} + i\hat{y}')/\hat{F}$ , equation (1) becomes

$$\mathbf{y}_1 = e^{-i\omega}(\mathbf{y}_0 + i s_0), \quad (2)$$

where  $s_0 = \text{sign}[\text{Re}(\mathbf{y}_0)] = \pm 1$  is the normalized force. Note that in the map (2) the localization of the electron cloud is responsible of the creation of several structure resonances. Differently from an axis-symmetric particle distribution, in this model the Coulomb force does not scale

as  $1/y$ , but similarly to the space charge the force is anti-symmetric with regard to the exchange  $y \leftrightarrow -y$ . The repeated application of the map (2) creates, for  $Q_y$  close to some structure resonance  $Q_r = m/n$ , a set of  $n$  fixed points according to the distance  $\delta$  from the resonance. The phase advance is  $\omega = 2\pi m/n + \delta$ . For  $m = 1$  one of these  $n$  fixed points, we call it  $\mathbf{y}_0^{(n)}$ , has the location

$$\mathbf{y}_0^{(n)} = \frac{e^{i[-\pi/2 - \omega/2]}}{2 \tan(n\delta/4) \sin(\omega/2)}. \quad (3)$$

The other  $(n-1)$  fixed points are found by repeatedly mapping  $\mathbf{y}_0^{(n)}$  with (2). In Fig. 2 we show the phase space

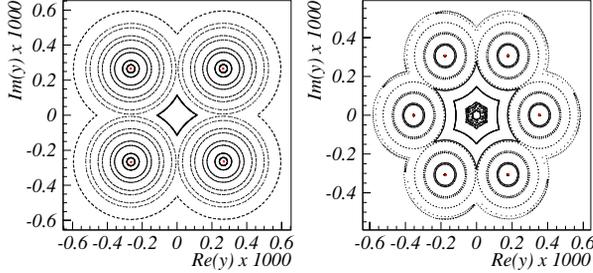


Figure 2: Phase space structure for  $Q_r = 1/4$  (left) and  $Q_r = 1/6$  (right). In both pictures  $\delta = -0.0003$  and  $\hat{R}_1 = 0$ .

plot for  $Q_y$  close to the resonances at  $Q_r = 1/4$  (left), and  $Q_r = 1/6$  (right). The distance from the resonance in both pictures is  $\delta = -0.0003$ . Note that the number of islands is even because of the antisymmetric character of the Coulomb force created by our model of EC: one can prove that the orbit created by an initial point  $\mathbf{y}_0 \rightarrow O(\mathbf{y}_0)$  and the orbit created by the initial point  $\exp(i\pi)\mathbf{y}_0 \rightarrow O(\exp(i\pi)\mathbf{y}_0)$  are the same except for a rotation by  $\pi$ . This property, which stems from the symmetry of the map (2), is responsible for creating island chains that are multiples of 2. For example, for  $Q_r = 1/3$  this property implies that the real resonance is  $Q_r = 2/6$  and that the phase space will be formed by 2 disconnected sets of 3-islands. From (2) it is easy to show that the secondary tune is  $\omega_{sec} = \delta$ , and, observing that the islands are circular, the island size, i.e. its radius, is

$$S_0^{(n)} = \frac{1}{2|\tan(n\delta/4)|}. \quad (4)$$

When the proton particle is located at a different (later) longitudinal position, the EC structure of the pinch is met at a different stage of its development. In the electron cloud stripe map this is included by changing the EC force in (1) to  $\mathcal{F}(\hat{y}_0) = \hat{F} \text{sign}[Re(\hat{y}_0)] \Theta(|\hat{y}_0|/\hat{R}_1 - 1)$ . This expression takes into account that inside the two EC planes the force is zero and outside of them the force is constant. In Fig. 3 we show the effect of the 2 EC planes when  $\hat{R}_1$  is 1/8-th of the island size in Fig. 2. The green lines indicate  $\pm\hat{R}_1$ , i.e. the locations of the EC planes. Although the region between the green lines has no electric field, the dynamics created by the new force  $\mathcal{F}$  brings the particles

into and out of this field-free region. The integrated effect results in the creation of a new island chain as visible in Fig. 3. The size of these new islands is  $\hat{R}_1$ .

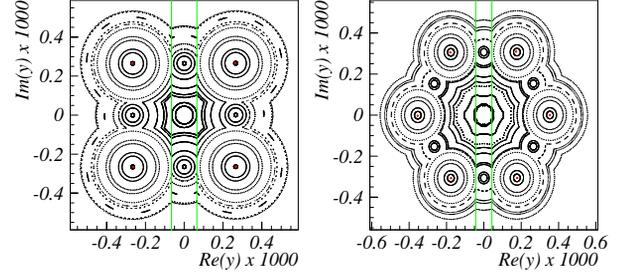


Figure 3: Phase space for  $Q_r = 1/4$  (left) and  $1/6$  (right). The distance of the EC planes,  $\hat{R}_1$ , is 1/8th of the island radius in Fig. 2, and  $\delta = -0.0003$ .

### RESONANCE CROSSING

At the start of the pinch no electrons are accumulated and the strength of the electric field is  $\hat{F} = 0$ . During the build up of the pinch  $\hat{F} (< 0)$  grows along the bunch, from zero up to a maximum value, and then it remains constant (in this modeling). Let us assume that  $\hat{F}' = d\hat{F}/dN = \text{const.}$ , with  $N$  denoting the turn number. Due to the increase of  $\hat{F}$  the fixed points created by a structure resonance  $Q_r = m/n$  migrate outwards according to

$$\frac{d|\hat{\mathbf{y}}_0^{(n)}|}{dN} = \frac{\hat{F}'}{2 \tan(n\delta/4) \sin(\omega/2)}. \quad (5)$$

Here  $\hat{\mathbf{y}} = \hat{y} + i\hat{y}' = \mathbf{y}\hat{F}$ . The moving islands crossing a particles orbit will trap the latter if: *The speed of rotation of the particle in the islands is faster than the speed of escape of the fixed points.* The maximum speed of a particle in the islands is

$$v_p = |\omega_{sec}| \hat{S}_0^{(n)} = \frac{|\delta|}{2} \frac{\hat{F}}{\tan(n\delta/4)}. \quad (6)$$

Detrapping [6] occurs if the opposite condition is fulfilled, i.e. if  $v_p < d|\hat{\mathbf{y}}_0^{(n)}|/dN$ . The transition can be characterized by a dimensionless parameter  $T$  defined as the ratio of these two speeds

$$T = \frac{d|\hat{\mathbf{y}}_0^{(n)}|}{dN} \frac{1}{|\omega_{sec}| \hat{S}_0^{(n)}}. \quad (7)$$

The condition of detrapping reads  $T > 1$ .  $T$  can be also interpreted as an adiabaticity parameter: for  $T \ll 1$  the motion of the fixed points is adiabatic with respect to the motion in the islands. The condition  $T = 1$  corresponds to an EC strength of  $\hat{F}_t = \hat{F}' / [|\delta| \sin(\omega/2)]$ . With  $\hat{F}' = d\hat{F}/dN = \text{const.}$  the previous formula yields the number of turns  $N$  at which the transition takes place:  $\Delta N_t = 1/[|\delta| \sin(\omega/2)]$ . At this transition instant the fixed points are located at  $|\hat{\mathbf{y}}_c^{(n)}| = \hat{F}' / \{2|\delta|[\tan(n\delta/4) \sin^2(\omega/2)]\}$ . If beam particles have amplitudes larger than  $|\hat{\mathbf{y}}_c^{(n)}|$  they will be trapped, otherwise scattered [7, 8]. However, when  $\hat{\mathbf{y}}_c^{(n)}$  is at  $T = 1$ , the part of the island closer to the origin

can still trap beam particles. Therefore, trapping does never occur only for those particles with  $|\hat{y}| < |\hat{y}_{dt}|$ , where

$$|\hat{y}_{dt}| = |\hat{y}_c^{(n)}|[1 - \sin(\omega/2)]. \quad (8)$$

For those trapped their maximum amplitude is  $|\hat{y}_{max}| = |\hat{F}_{max} \mathbf{y}_c^{(n)}|$ . In Fig. 4 we show the benchmarking of (8) with the map (1).

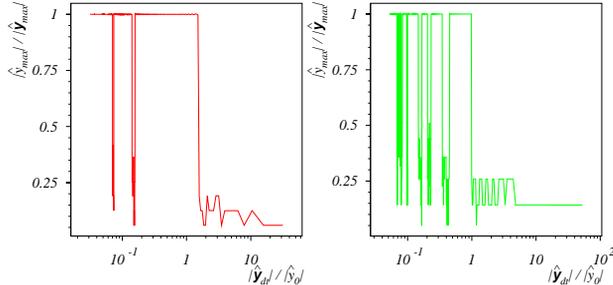


Figure 4: Benchmarking of (8). For particles with  $|\hat{y}_{dt}| > |\hat{y}_0|$  trapping does not take place and their amplitude is scattered. For  $|\hat{y}_{dt}| < |\hat{y}_0|$  particles are trapped and reach  $|\hat{y}_{max}|$ . The colors refer to the 4th (red) and 6th order (green) resonances.

### SCATTERING REGIME

From (8) trapped particles should have an amplitude larger than  $|\hat{y}_{dt}| \propto \hat{F}'/\delta$ .  $\hat{F}'$  depends on the synchrotron tune, on the longitudinal particle amplitude and on the extent of the pinch along the longitudinal axis ( $z$ ). As the latter is much shorter than the bunch length it is likely that the scattering regime is dominating the EC periodic crossing dynamics. We here introduce an analysis of the scattering regime. The beam dynamics in the scattering regime ( $T > 1$ ) is characterized by the absence of trapping [9, 10, 11]. This means that when the resonant orbit crosses the particle orbit, the CS invariant of the particle is subjected to a random kick of maximum amplitude equal to the island size. If we characterize the particle via its radius  $r_0$  (proportional to the root square of the action), we can represent the scattering process via the map

$$r_1 = r_0 + \xi \frac{\pi}{n} r_0, \quad (9)$$

where  $r_1$  is the new particle radius after one crossing. In (9) the quantity  $\pi/nr_0$  is the radius of an island, and  $\xi$  is a random variable of standard deviation  $\sigma_\xi$ , the value of which incorporates the effectiveness of the island to scatter the particle radius (invariant) at  $r_0$ . For  $T \gg 1$  the islands cross the particle orbit very fast producing a small scattering, hence  $\sigma_\xi \rightarrow 0$ . The general dependence is a complex function of  $T$ , like  $\xi = \xi(1/T)$ . However, as a first approach for understanding the problem, we assume  $\xi$  to be independent of  $T$ : this is the case when  $1/T$  is close to 1, as there the scattering involves the full island. Equation (9) then describes then the so called *Geometric Brownian Motion*, and is equivalent to the stochastic equation  $dr = \sigma_\xi r dW$ , where  $W$  is a Wiener process. The correspondent Fokker-Planck equation reads

$$\frac{\partial f}{\partial t} = \frac{1}{2} \sigma_\xi^2 \frac{\partial^2}{\partial r^2} (r^2 f), \quad (10)$$

where  $f$  is the beam distribution. If the beam pipe is taken at  $r_p$  we can define the rescaled quantities  $(Z, \tau)$  via  $r = r_p Z$ , and  $t = (2\sigma_\xi^{-2})\tau$ . By taking an initially uniform beam distribution in Courant-Snyder coordinates  $(\hat{y}, \hat{y}')$  of radius  $A$ , we integrate over the boundary  $0 < Z < a (< 1)$ , with  $a = A/r_p$  and obtain

$$f(Z, \tau) = \frac{4\sqrt{a}}{5\pi} \int_0^\infty e^{-(\beta^2+1/4)\tau} \times \frac{\sin(-\beta \ln a) + (2\beta/5) \cos(-\beta \ln a)}{1+(2\beta/5)^2} \frac{\sin(\beta \ln Z)}{Z^{3/2}} d\beta. \quad (11)$$

The integration of  $f(Z, \tau)$  over  $Z$  yields the beam survival  $N(\tau) = \int_0^1 f(Z, \tau) dZ$ . Figure 5 show a benchmarking of the solution (11) with the numerical simulation of (9).

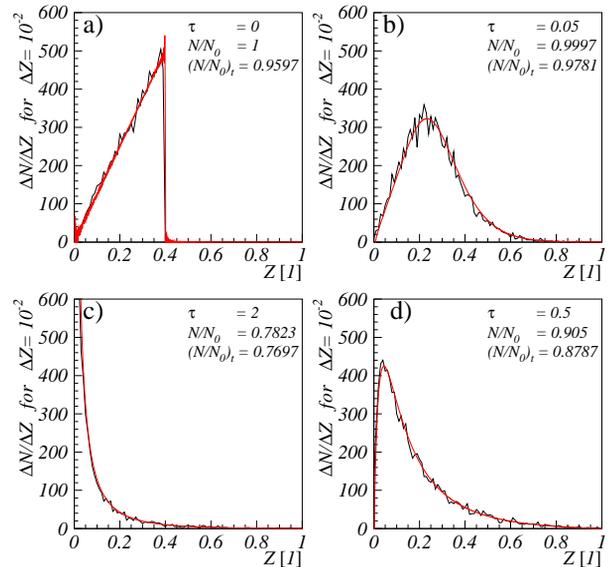


Figure 5: Evolution of a KV beam distribution driven by geometric Brownian motion from simulation (black curve), and the (numerically integrated) analytic solution (red). The pictures a), b), c), and d) are obtained at different instants during the beam evolution. The simulation used  $10^4$  macro-particles. In a) the theoretical initial value is not 1, but 0.95 due to the approximations of the numerical integration.

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