LONGITUDINAL DRIFT COMPRESSION OF INTENSE CHARGED PARTICLE BEAMS*

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Abstract

To achieve high focal spot intensities in ion-beam-driven high energy density physics and heavy ion fusion applications, the ion beam must be compressed longitudinally by factors of ten to one hundred before it is focused onto the target. The longitudinal compression is achieved by imposing an initial velocity profile tilt on the drifting beam, and allowing the beam to compress longitudinally until the space-charge force or the internal thermal pressure stops the longitudinal compression of the charge bunch. In this paper, the problem of longitudinal drift compression of intense charged particle beams is analyzed analytically using a one-dimensional warm-fluid model describing the longitudinal beam dynamics. The hodograph transformation is used to transform the nonlinear fluid equations into a single, second-order, linear partial differential equation (PDE). The approximate general solution of this equation describing the intense beam system with stagnation point is obtained.

THEORETICAL MODEL

In the present analysis, we employ a one-dimensional warm-fluid model [1, 2] with adiabatic equation of state ($ds/dt = 0$, where s is the entropy per unit volume) to describe the longitudinal nonlinear beam dynamics with average electric field given by the g-factor model with $e_b E_z = -e_b^2 g \partial\lambda/\partial x$ [3]. For example, for a transversely-space-charge-dominated beam with flat-top density profile in the transverse plane, $g \simeq 2\ln(r_w/r_b)$ [4, 3] while for a transversely-emittance-dominated beam $g \simeq 2\ln(r_w/r_b) + \alpha$ where the numerical coefficient α depends on the transverse beam-density profile. Here, $\lambda(x,t)$ is the line density, e_b is the charge of a beam particle, r_w is the conducting wall radius, and r_b is the beam radius. Generally, the beam radius, and therefore the g-factor, are functions of the line density and the external transverse focusing, and can change during the beam compression. Here we consider only the case of a transversely-emittance-dominated beam when the beam radius, and therefore the g-factor, remain constant during compression.

The macroscopic fluid equations for the line density $\lambda(x,t)$, the average longitudinal beam velocity $v(x,t)$, and the longitudinal line pressure $p(x,t)$ are given by [1, 2]

$$\frac{\partial\lambda}{\partial t} + \frac{\partial}{\partial x}(\lambda v) = 0, \tag{1}$$

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\frac{v^2}{2} = -\frac{e_b^2 g}{m_b}\frac{\partial\lambda}{\partial x} - \frac{1}{m_b\lambda}\frac{\partial p}{\partial x} = -\frac{\partial w}{\partial x}, \tag{2}$$

where $p = (p_0/\lambda_0^3)\lambda^3$ for a triple-adiabatic equation-of-state. Here we have introduced the effective potential w defined by

$$w \equiv c_g^2\frac{\lambda}{\lambda_0} + \frac{c_p^2}{2}\frac{\lambda^2}{\lambda_0^2}, \tag{3}$$

where $c_g^2 = e_b^2 g\lambda_0/m_b$ and $c_p^2 = 3p_0/m_b\lambda_0$ are constants with dimensions of $(\text{speed})^2$, m_b is the mass of a beam particle, and λ_0 and p_0 are constants with the dimensions of line density and line pressure, respectively.

In the remainder of this section we summarize the well-established theoretical technique developed in fluid mechanics [5] that can be used to solve the nonlinear fluid equations (1) and (2). By introducing the velocity potential ϕ, where $v = \partial\phi/\partial x$, we can rewrite Eq. (2) as

$$\frac{\partial\phi}{\partial t} + \frac{v^2}{2} + w = 0. \tag{4}$$

The full differential of ϕ then becomes

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial t}dt = v dx - \left(\frac{v^2}{2} + w\right)dt. \tag{5}$$

Introducing $\chi = \phi - xv + t(w + v^2/2)$, Eq. (5) can be expressed as

$$d\chi = -x dv + t d\left(\frac{v^2}{2} + w\right) = t\frac{dw}{d\lambda}d\lambda + (vt - x)dv. \tag{6}$$

It follows from Eq. (6) that χ can be considered as a function of the new independent variables (v, λ), and that

$$t = \frac{1}{(dw/d\lambda)}\frac{\partial\chi}{\partial\lambda}, \quad \text{and} \quad x - vt = -\frac{\partial\chi}{\partial v}. \tag{7}$$

Therefore, if the function χ is known as a function of its arguments (v, λ), then Eq. (7) gives (v, λ) as implicit functions of (x, t).

Next we apply the same approach to solving the Eq. (1). By introducing the potential $\bar\phi$, where $\lambda = \partial\bar\phi/\partial x$, we can rewrite Eq. (1) as

$$\frac{\partial\bar\phi}{\partial t} + v\lambda = 0. \tag{8}$$

The full differential of $\bar\phi$ then becomes

$$d\bar\phi = \frac{\partial\bar\phi}{\partial x}dx + \frac{\partial\bar\phi}{\partial t}dt = \lambda dx - v\lambda dt. \tag{9}$$

Introducing $\bar\chi = \bar\phi - x\lambda + tv\lambda$, it follows that $d\bar\chi$ can be expressed as

$$d\bar\chi = -x d\lambda + t d(v\lambda) = t\lambda dv + (vt - x)d\lambda. \tag{10}$$

It follows from Eq. (10) that $\bar\chi$ can be considered as a function of the new independent variables (v, λ), and that

$$t = \frac{1}{\lambda}\frac{\partial\bar\chi}{\partial v}, \quad \text{and} \quad x - vt = -\frac{\partial\bar\chi}{\partial\lambda}. \tag{11}$$

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Therefore, if the function $\bar{\chi}$ is known as a function of its arguments (v, λ), then Eq. (11) gives (v, λ) as implicit functions of (x, t).

It follows from Eqs. (7) and (11) that the functions χ and $\bar{\chi}$ are related as

$$\frac{\partial}{\partial \lambda}\chi = \frac{1}{\lambda}\frac{dw}{d\lambda}\frac{\partial}{\partial v}\bar{\chi}, \quad \frac{\partial}{\partial \lambda}\bar{\chi} = \frac{\partial}{\partial v}\chi. \tag{12}$$

Combining Eqs. (12), we obtain the equation for $\bar{\chi}$ [5]

$$\lambda\frac{\partial^2}{\partial \lambda^2}\bar{\chi} = \frac{dw}{d\lambda}\frac{\partial^2}{\partial v^2}\bar{\chi}. \tag{13}$$

Note that Eq. (13) is a linear partial differential equation for the function $\bar{\chi}(v, \lambda)$. Equation (13), together with Eq. (11) can be used to obtain the solution to the system of equations (1) and (2) everywhere in the (x, t) plane except in the regions corresponding to *simple wave* solutions where $v = v(\lambda)$ [5].

GENERAL SOLUTION

Next we obtain a general solution of Eq. (13). Introducing the scaled density variable $\bar{\lambda} = \lambda/\lambda_0$, Eq. (13) can be expressed as

$$\bar{\lambda}\frac{\partial^2}{\partial \bar{\lambda}^2}\bar{\chi} = (c_g^2 + c_p^2\bar{\lambda})\frac{\partial^2}{\partial v^2}\bar{\chi}. \tag{14}$$

To determine the general solution to Eq. (14), we first Fourier transform with respect to the v dependence. This gives

$$\bar{\lambda}\frac{d^2\bar{\chi}_k}{d\bar{\lambda}^2} + (c_g^2 + c_p^2\bar{\lambda})k^2\bar{\chi}_k = 0. \tag{15}$$

Equation (15) is linear differential equation with linear coefficients and can be solved using the Laplace method [6]. Using this method, the solution can be expressed as the complex integral

$$\bar{\chi}_k(\bar{\lambda}) = \int_C dpV_k(p), \tag{16}$$

where the integration contour C connects two points p_1 and p_2 in the complex plane p, and $V_k(p_1) = V_k(p_2)$ where function $V_k(p)$ is defined by

$$V_k(p) = \exp\left\{ik\left(\frac{c_p\bar{\lambda}}{\tanh(p)} + \frac{c_g^2}{c_p}p\right)\right\}, \tag{17}$$

Using Eqs. (16) and (17) and performing the inverse Fourier transform, the general solution to Eq. (14) can be expressed as

$$\bar{\chi}(v, \bar{\lambda}) = \int_0^\infty dp\left\{f_1\left[\left(\frac{c_p\bar{\lambda}}{\tanh(p)} + \frac{c_g^2}{c_p}p\right) - v\right] + f_2\left[\left(\frac{c_p\bar{\lambda}}{\tanh(p)} + \frac{c_g^2}{c_p}p\right) + v\right]\right\}, \tag{18}$$

where f_1 and f_2 are arbitrary functions such that the integrals in Eq. (18) converge. Alternatively we can use Eq. (12) to find the general solution for the function $\chi(v, \bar{\lambda})$, i.e.,

$$\chi(v, \bar{\lambda}) = \int_0^\infty \frac{dp}{\tanh(p)}\left\{q_1\left[\left(\frac{c_p\bar{\lambda}}{\tanh(p)} + \frac{c_g^2}{c_p}p\right) - v\right] + q_2\left[\left(\frac{c_p\bar{\lambda}}{\tanh(p)} + \frac{c_g^2}{c_p}p\right) + v\right]\right\}, \tag{19}$$

where q_1 and q_2 are arbitrary functions such that the integrals in Eq. (19) converge.

GENERAL SOLUTION OF THE INITIAL VALUE PROBLEM

In this section we make use of Eq.(19) to solve the initial-value problem for the case of beam expansion into vacuum. The initial conditions for this problem are zero flow velocity at every point, $v_0(x, 0) = 0$, and prescribed density profile $\lambda(x, 0) = \lambda_0(x)$, which expresses the initial line density as a function of x. At some later time $t = t_f$, the density and velocity profiles will be given by the functions $\lambda(x, t_f)$ and $v(x, t_f)$ which are the solutions to Eqs. (1) and (2). Since the equations of motion [Eqs. (1) and (2)] are time-reversible, the flow described by $\bar{\lambda}(x, t) = \lambda(x, t_f - t)$ and $\bar{v}(x, t) = -v(x, t_f - t)$ are also solutions to these equations with initial conditions $\bar{v}(x, 0) = -v(x, t_f)$ and $\bar{\lambda}(x, 0) = \lambda(x, t_f)$. At time $t = t_f$ this flow has zero velocity profile ($\bar{v} = 0$) and the density profile is given by the initial profile for the expansion problem, i.e., $\bar{\lambda}(x, t_f) = \lambda_0(x)$.

To solve the initial-value problem, we assume that the density profile $\lambda_0(x)$ decreases monotonically to zero at the beam boundary $x = \pm x_0$, is an even function of x, and is an invertable function for $x > 0$ everywhere where the density is non-zero. Therefore, we assume that at $t = 0$ the inverted profile $x_0(\lambda)$ is known. The condition that $\lambda_0(x)$ decreases monotonically to zero at the beam boundary means that no rarefaction wave is launched from the boundary into the beam as it expands. Since we are interested in the time-reversed problem of beam compression, we assume that multi-valued flow does not form as the beam expands. This is equivalent to considering only initial density profiles with first derivative decreasing continuously from the beam center to the beam edge. This guarantees that the portions of the beam with smaller density accelerate faster than the portions with larger density, and as a result, the flow is never multi-valued.

In general, there are four regions of flow, and each is separated from the others by two characteristics. The region of flow in the (x, t) plane and its boundaries are illustrated Ref. [7]. There are simple relations connecting flows in all regions with the flow in region I, where $v = 0$ at $t = 0$ [7], which will be analyzed here in more detail.

To satisfy the initial condition $v = 0$ at $t = 0$, we are required to choose

$$\chi^I(v,\bar\lambda) = \frac{1}{2}\int_0^\infty \frac{dp}{\tanh(p)}\left\{q\left[\left(\frac{c_p\bar\lambda}{\tanh(p)}+\frac{c_g^2}{c_p}p\right)\right.\right.$$
$$\left.\left.+v\right]-q\left[\left(\frac{c_p\bar\lambda}{\tanh(p)}+\frac{c_g^2}{c_p}p\right)-v\right]\right\}. \quad (20)$$

Substituting Eq. (20) into Eq. (7) at $t=0$, we obtain

$$x_0(\bar\lambda) = -\int_0^\infty \frac{dp}{\tanh(p)}q'\left[\left(\frac{c_p\bar\lambda}{\tanh(p)}+\frac{c_g^2}{c_p}p\right)\right]. \quad (21)$$

To invert equation (21), we introduce the new variable $\xi = n(y)/n(y^*)$, where the function n is to be specified later. Here $y=(c_g^2/c_p)p+c_p\bar\lambda/\tanh(p)$, and $y^*=(c_g^2/c_p)p^*+c_p\bar\lambda/\tanh(p^*)$ is the minimum value of $y(p)$ reached at $p=p^*$, where $\sinh^2(p^*)=(c_p/c_g)^2\bar\lambda$.

The solutions in Eqs. (20) and 21) can be expressed as

$$\chi^I(v,\bar\lambda) = \frac{1}{2}\int_1^\infty d\xi S(\xi,y^*)\left\{q\left[n^{-1}(\xi n(y^*))+v\right]\right.$$
$$\left.-q\left[n^{-1}(\xi n(y^*))-v\right]\right\}, \quad (22)$$

$$\bar x_0(y^*) = -\int_1^\infty d\xi S(\xi,y^*)q'\left[n^{-1}(\xi n(y^*))\right], \quad (23)$$

where $\bar x_0(y^*)\equiv x_0(\bar\lambda)$, and the function $S(\xi,y^*)$ is given by

$$S(\xi,y^*) = \frac{1}{2}\frac{n(y^*)}{n'(y)}\left[\frac{\sinh(2p)}{\sinh^2(p)-(c_p/c_g)^2\bar\lambda}\right.$$
$$\left.-\frac{\sinh(2\bar p)}{\sinh^2(\bar p)-(c_p/c_g)^2\bar\lambda}\right], \quad (24)$$

where $\bar p \le p^* \le p$ is the root of the equation $p+(c_p/c_g)^2\bar\lambda/\tanh(p) = \bar p+(c_p/c_g)^2\bar\lambda/\tanh(\bar p)$. Next, we note that for a particular choice of function $n(y)$, the function $S(\xi,y^*)$ is a function of only one variable ξ, and is independent on y^*. Here, we present only an approximate solution by choosing $n(y)=\exp[(c_p/c_g^2)y/2]-1$. For this choice, the function $S(\xi,y^*)$ is given approximately by $S(\xi,y^*)\approx 2/\sqrt{\xi^2-1}$, with an accuracy of about 5%.

Equation (23) can now be inverted by using the integral Abel transform. This gives

$$\chi^I(v,\bar\lambda) = \int_1^\infty \frac{d\xi}{\sqrt{\xi^2-1}}\left\{q\left[n^{-1}(\xi n(y^*))+v\right]\right.$$
$$\left.-q\left[n^{-1}(\xi n(y^*))-v\right]\right\}, \quad (25)$$

$$q(y) = \frac{1}{\pi}\int_y^{z_{max}}\bar x_0(z)dz\frac{d}{dz}\int_y^z \frac{dt}{\left[\frac{n^2(z)}{n^2(t)}-1\right]^{1/2}}, \quad (26)$$

where

$$z_{max} = c_g\left\{\left(1+\frac{c_p^2}{c_g^2}\right)^{1/2}+\frac{c_g}{c_p}\ln\left[\frac{c_p}{c_g}+\left(1+\frac{c_p^2}{c_g^2}\right)^{1/2}\right]\right\}, \quad (27)$$

In the limit of a cold beam with $c_p\to 0$, Eqs. (25) and (26) reduce to

$$q(2c) = \frac{1}{\pi}\int_c^{c_0}\frac{\bar c x_0(\bar c)d\bar c}{\sqrt{\bar c^2-c^2}}, \quad (28)$$

and

$$\chi^I(v,\bar\lambda) = \int_1^\infty \frac{d\xi}{\sqrt{\xi^2-1}}\{q[2\xi c+v]-q[2\xi c-v]\}, \quad (29)$$

where $c=c_g\bar\lambda^{1/2}$ is the speed of sound in the limit $c_p\to 0$.

In the limit of negligible space charge with $c_g\to 0$, Eqs. (25) and (26) reduce to

$$q(c) = \left(\frac{c_g^2}{c_p}\right)x_0(c), \quad (30)$$

and

$$\chi^I(v,\bar\lambda) = \frac{1}{2}\int_0^\infty d\eta\,[x_0(\eta+c+v)-x_0(\eta+c-v)]$$
$$= \frac{1}{2}\int_{c+v}^{c-v} d\bar c x_0(\bar c), \quad (31)$$

where $c=c_p\bar\lambda$ is the speed of sound in the limit $c_g\to 0$.

Equations (28) and (29) for $c_p\to 0$, and Eq. (31) for $c_g\to 0$, are identical to the equations obtained previously in Ref. [7] for these two limiting cases.

CONCLUSIONS

To summarize, we have studied the longitudinal drift compression of an intense charged particle beam using a one-dimensional warm-fluid model. We have reformulated the drift compression problem as the time-reversed expansion problem of the beam with arbitrary line density profile and zero velocity profile. We have obtained approximate (within 5% accuracy) analytical solutions to the expansion problem using a general formalism, which reduces the system of warm-fluid equations to a linear second-order partial differential equation.

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