UNCERTAINTY MODELLING OF RESPONSE MATRIX

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Abstract

Electron orbit feedback controllers are based on the inversion of the response matrix of the storage ring and as a result, mismatches between the accelerator model and the real machine can limit controller performance or cause the controller to become unstable. In order to perform stability analysis tests of the controller, accurate uncertainty descriptions are required. In this paper, BPM scaling errors, actuator scaling errors and drifts in tune are considered as the main sources of spatial uncertainties and because most electron orbit feedback systems use Singular Value Decomposition (SVD) to decouple the inputs and outputs of the system, the uncertainty can be expressed in terms of this decomposition. However SVD does not allow the main sources of uncertainty to be decoupled so instead, a Fourier-based decomposition of the response matrix is used to decouple and model the uncertainties. In this paper, both Fourier and SVD uncertainty modelling methods are applied to the Diamond Light Source storage ring and compared.

INTRODUCTION

The response matrix, $R \in \mathbb{R}^{M \times N}$ is the steady state (DC) response of the correctors to a change in the transverse orbit position measured at the m^{th} beam position monitors (BPMs) due to a transverse kick at the n^{th} corrector. For correction of the electron orbit, a map from BPM to corrector is required and therefore the inverse of the nominal (or "golden") response matrix is required. The inverse is normally computed using a Singular Value Decomposition (SVD) approach, such that

$$\begin{aligned} R_o &= \Phi_o \Sigma_o \Psi_o^{\mathrm{T}} \\ R_o^{-1} &= \Psi_o \Sigma_o^{-1} \Phi_o^{\mathrm{T}}. \end{aligned} \tag{1}$$

However because of BPM and corrector scaling errors from imperfect calibrations or drifts in the tune, the actual response of the machine may differ over time from the ideal response. This means that the control action calculated from the ideal response may be either too aggressive or too conservative. In the case of such model mismatch, applying the control action may result in closed loop instabilities or even if the closed loop remains stable, steady state performance will degrade. By modelling the various sources of the uncertainties, the mismatch between the machine response and the ideal response can be quantified and the uncertainty description can then be used in stability analysis techniques to determine the allowable bound on the size of uncertainty to ensure closed loop stability ([1] in these proceedings). In this paper, modelling of the uncertainties associated with the response matrix is presented. Firstly, the uncertainties are described within the commonly used SVD framework. However there are limitations to this approach and as an alternative, a harmonic decomposition of the response matrix is presented and the uncertainty is expressed in terms of the Fourier coefficients of the response.

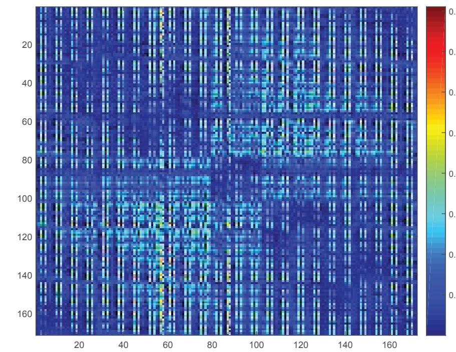

Figure 1: Additive uncertainty in response matrix R_0 ($|\Delta_R|$).

UNCERTAINTY DESCRIPTIONS USING SINGULAR VALUE DECOMPOSITION

The uncertainty in the response matrix can be expressed by an additive error,

$$R = R_o + \Delta_R \tag{2}$$

where R represents a measured response matrix and R_o represents the nominal response matrix. Figure 1 shows the structure of Δ_R for the Diamond storage ring taken from one measurement of R. The uncertainty in the response matrix combines the errors in Φ, Σ and Ψ and it is not straightforward to distinguish, for example, the effect of BPM errors from corrector errors. It is therefore more useful to model the uncertainty in the process output (sensor values) by multiplicative operators [2], which can be expressed as

$$R = (I + \Delta_B)R_0 \tag{3}$$

so that

$$\Delta_B = RR_0^{-1} - I. \tag{4}$$

The uncertainty can be projected into "mode space", such that

$$\Delta_B = \Phi\Sigma\Psi^{\mathrm{T}}\Psi_0\Sigma_0^{-1}\Phi_0^{\mathrm{T}} - I. \tag{5}$$

Likewise, the same treatment can be given to uncertainties in the process input (corrector values), which is expressed as

$$R = R_0(I + \Delta_C) \tag{6}$$

so that

$$\Delta_C = \Psi_0\Sigma_0^{-1}\Phi_0^{\mathrm{T}}\Phi\Sigma\Psi^{\mathrm{T}} - I. \tag{7}$$

By assuming that all the uncertainty lies in either Φ or Ψ, then the uncertainties can be expressed as

$$\begin{aligned} \bar{\Delta}_B &= \Phi\Phi_0^{\mathrm{T}} - I \\ \bar{\Delta}_C &= \Psi_0\Psi^{\mathrm{T}} - I. \end{aligned} \tag{8}$$

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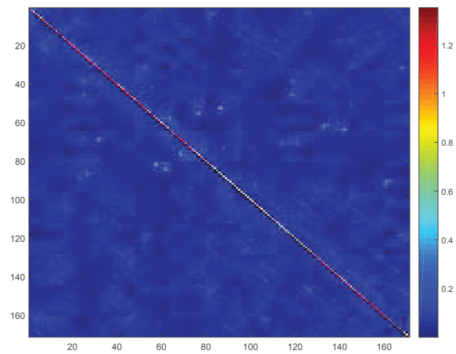

Figure 2: Multiplicative uncertainty in left singular vectors, Φ_0 ($\bar{\Delta}_B$).

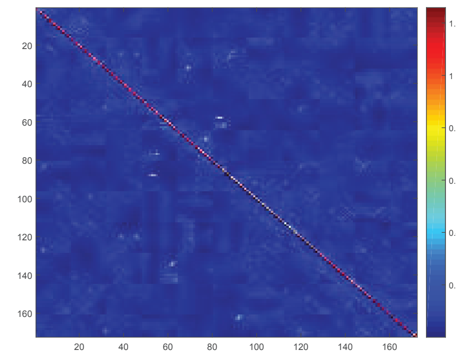

Figure 3: Multiplicative uncertainty in right singular vectors, Ψ_0 ($\bar{\Delta}_C$).

The shape of these uncertainties are shown in Fig. 2 and Fig. 3 respectively where the uncertainty between modes are the most significant (represented by the diagonal elements).

The uncertainty in the singular values can also be included as a multiplicative error, such that

$$\Delta_\Sigma = \Sigma_0^{-1}\Sigma - I \tag{9}$$

where Δ_Σ is diagonally structured since Σ_0 and Σ are diagonal. The diagonal values are shown in Fig. 4 where the size of the uncertainty in the higher order modes (associated with small singular values) is greater than the error in low order modes. Therefore the high order modes are more vulnerable to stability problems.

The response matrix with all sources of uncertainty can then be written as

$$R = (I + \bar{\Delta}_B)\Phi_0 \;\; \Sigma_0(I + \Delta_\Sigma) \;\; \Psi_0(I + \bar{\Delta}_B) \tag{10}$$

While the uncertainty descriptions using the SVD matrices are useful for analysis, such as identifying which modes are more affected by uncertainties, there is no physical interpretation of the uncertainty structure and effects from individual BPMs and correctors are difficult to decouple. Instead, a harmonic decomposition method is proposed in the next section so that the effects of BPMs, corrector and tune drift can be decoupled and modelled.

UNCERTAINTY DESCRIPTIONS USING HARMONIC DECOMPOSITION

Harmonic Decomposition of the Response Matrix

For a transverse excitation θ_n at the n^{th} corrector and measured by the m^{th} BPM, the disturbed closed orbit may be expressed as

$$y_m = \frac{\theta_c\sqrt{\beta_m}\sqrt{\beta_n}}{2\sin\pi\nu}\cos\left(\nu\pi - (|\eta_m - \eta_n|)\right) \tag{11}$$

where β_m and β_n are the magnitudes of the beta function at BPM and corrector locations respectively and η_m and η_n are the phase advances at BPM and corrector locations respectively and ν is the tune. By introducing the Courant-Snyder variables [3]

$$\tilde{y}_m = \frac{y_m}{\sqrt{\beta_m}}, \quad \tilde{\eta} = \frac{\eta}{\nu} \tag{12}$$

Eq. 11 can be written as

$$\tilde{y}_m = \frac{\theta_n\sqrt{\beta_n}}{2\sin\pi\nu}\cos\nu\left(\pi - (|\tilde{\eta}_m - \tilde{\eta}_n|)\right). \tag{13}$$

Fourier analysis of Eq. 13 gives [4, 5]

$$\tilde{y}_m = \frac{\theta_n\beta_n}{2\sin\pi\nu}\sum_{f=0}^{\infty}\hat{\sigma}_f\cos f(\tilde{\eta}_m - \tilde{\eta}_n) \tag{14}$$

where

$$\begin{aligned}\hat{\sigma}_f &= \frac{(-1)^f}{\pi}\int_{-\pi}^{\pi}\cos f\tilde{\eta}\cos\nu\tilde{\eta}\,\mathrm{d}\tilde{\eta}\\ &= \frac{2\nu}{\nu^2 - f^2}\frac{\sin\nu\pi}{\pi} \quad (f = 1, 2, \ldots).\end{aligned} \tag{15}$$

The expression in Eq. 14 can be further simplified by

$$\tilde{y}_m = \theta_n\beta_n\sum_{f=0}^{\infty}\hat{\sigma}_f\cos f(\tilde{\eta}_m - \tilde{\eta}_n) \tag{16}$$

where the Fourier coefficients are redefined as

$$\hat{\sigma}_f = \frac{1}{2\pi}\frac{2\nu}{\nu^2 - f^2} \quad (f = 0, 1, 2, \ldots). \tag{17}$$

Therefore each element of the response matrix is defined as y_m/θ_n is given by

$$\begin{aligned}r_{mn} &= \sqrt{\beta_m}\sqrt{\beta_n}\sum_{f=-F}^{F}\hat{\sigma}_f\cos f(\tilde{\eta}_m - \tilde{\eta}_n)\\ &= \sqrt{\beta_m}\sqrt{\beta_n}\sum_{f=-F}^{F}\hat{\sigma}_f\mathrm{Re}(\mathrm{e}^{if\tilde{\eta}_m}\mathrm{e}^{-if\tilde{\eta}_n})\end{aligned} \tag{18}$$

for $f = [-F, \ldots, F]$ where F is chosen (without loss of generality) so that $F \leq M$ where $M \leq N$. In matrix form

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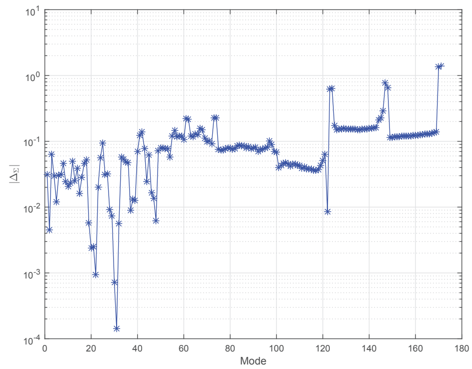

Figure 4: Multiplicative uncertainty in singular vectors, Σ_0 ($|\Delta_\Sigma|$).

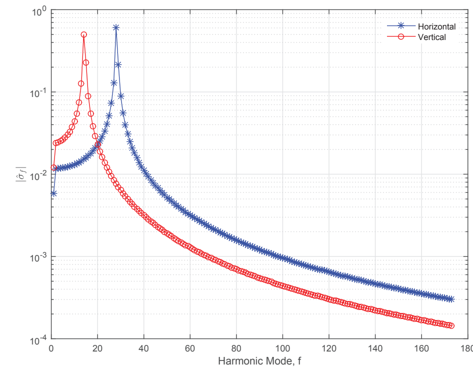

Figure 5: Horizontal ($*$ blue) and vertical ($\circ$ red) Fourier coefficients $|\hat{\sigma}_f|$ with vertical tune $\nu_y = 13.324$ and $\nu_x = 27.266$.

Eq. 18 becomes

$$R_0 = \hat{\Phi}_0 \hat{\Sigma}_0 \hat{\Psi}_0^{-\mathrm{T}} \tag{19}$$

where

$$\begin{aligned} \hat{\Sigma}_0 &= \mathrm{diag}_{f=-F,\dots,F}\{\hat{\sigma}_f\} \\ \hat{\Phi}_0 &= \mathrm{diag}_{m=1,\dots,M}\{\sqrt{\beta_m}\}\mathrm{Re}\left(\left[\mathrm{e}^{if\tilde{\eta}_m}\right]_{mf}\right) \\ \hat{\Psi}_0 &= \mathrm{diag}_{n=1,\dots,N}\{\sqrt{\beta_n}\}\mathrm{Re}\left(\left[\mathrm{e}^{-if\tilde{\eta}_n}\right]_{nf}\right) \end{aligned} \tag{20}$$

and $\hat{\Phi}_0 \in \mathbb{R}^{M\times(2F+1)}$, $\hat{\Sigma}_0 \in \mathbb{R}^{(2F+1)\times(2F+1)}$ and $\hat{\Psi}_0 \in \mathbb{R}^{N\times(2F+1)}$ [5]. The Fourier coefficients defined in Eq. 17 are completely determined by the tune and are shown in Fig. 5 for the vertical and horizontal response matrices of the storage ring. The resonance properties of the disturbed closed orbit are evident and show that the orbit is most sensitive to those Fourier components of the perturbation whose order is close to the tune. The left and right matrices $\hat{\Phi}_0$ and $\hat{\Psi}_0$ are determined by the betatron phases of the BPMs and corrector magnets, which in turn depends solely on the position of the BPMs and the strengths of the corrector magnets respectively. The columns of $\hat{\Phi}_0$ and $\hat{\Psi}_0$ are not orthogonal because of the uneven spacing of BPMs and corrector magnets. However, the matrices are block orthogonal reflecting the repetition of cells and symmetry of the ring.

Modelling Uncertainties using Harmonic Decomposition

Given the expression for Fourier coefficients in Eq. 17, a change in a single fourier coefficient with respect to the tune can be expressed as

$$\Delta_{\hat{\sigma}_f} \approx \left.\frac{\delta\hat{\sigma}_f}{\delta\nu}\right|_{\nu=\nu_0} \Delta_\nu \tag{21}$$

where ν_0 is the nominal tune and the partial derivative term can be written as

$$\left.\frac{\delta\hat{\sigma}_f}{\delta\nu}\right|_{\nu=\nu_0} = \frac{\hat{\sigma}_f}{\nu_0}\left(\frac{f^2+{\nu_0}^2}{f^2-{\nu_0}^2}\right). \tag{22}$$

Therefore, the uncertainty in a Fourier coefficient, $\Delta_{\hat{\sigma}_f}$ is dependent only on the tune and can be represented by a multiplicative operator

$$\Delta_{\hat{\sigma}_f} = \left(\frac{\hat{\sigma}_f}{\nu_0}\left[\frac{f^2+{\nu_0}^2}{f^2-{\nu_0}^2}\right]\right)\Delta_\nu \tag{23}$$

where Δ_ν represents the change in tune from the nominal value. In matrix form,

$$\Delta_{\hat{\Sigma}} = W_{\hat{\Sigma}}\Delta_\nu \tag{24}$$

where

$$W_{\hat{\Sigma}} = \mathrm{diag}\left\{\frac{\hat{\sigma}_f}{\nu_0}\left[\frac{f^2+{\nu_0}^2}{f^2-{\nu_0}^2}\right]\right\}. \tag{25}$$

Uncertainty can also be included in the left and right harmonic matrices as in the SVD approach, so a complete description of the spatially uncertain response can be written as

$$\begin{aligned} R &= \left(I+\Delta_{\hat{\Phi}}\right)\hat{\Phi}_0\ \hat{\Sigma}_0\left(I+\Delta_{\hat{\Sigma}}\right)\ \hat{\Psi}_0^{-\mathrm{T}}\left(I+\Delta_{\hat{\Psi}}\right) \\ &= \left(I+\Delta_{\hat{\Phi}}\right)\hat{\Phi}_0\ \hat{\Sigma}_0\left(I+W_{\hat{\Sigma}}\Delta_\nu\right)\ \hat{\Psi}_0^{-\mathrm{T}}\left(I+\Delta_{\hat{\Psi}}\right) \end{aligned} \tag{26}$$

where $\Delta_{\hat{\Sigma}}$ is a diagonal matrix with elements determined by Eq. 23, $\Delta_{\hat{\Phi}}$ represents errors in the BPM positions and $\Delta_{\hat{\Psi}}$ represents variations in the strengths of the corrector magnets.

Uncertainty Analysis

By introducing a constant energy deviation of 0.03 GeV, the beta function and phase advance changes, such that the resulting uncertainties in $\hat{\Phi}$ and $\hat{\Psi}$ associated with these changes are shown in Fig. 6 and Fig. 7. Because $\hat{\Phi}$ and $\hat{\Psi}$ are determined by the beta function and phase advance, both $\Delta_{\hat{\Phi}}$ and $\Delta_{\hat{\Psi}}$ have a large diagonal structure but coupling between the beta function and phase advance within a cell results in the off-diagonal elements being non-zero. However, each element of $\hat{\Phi}$ and $\hat{\Psi}$ is associate with beam parameters η and β at BPM and corrector locations respectively. Likewise the energy deviation causes a 0.2 % change in tune which

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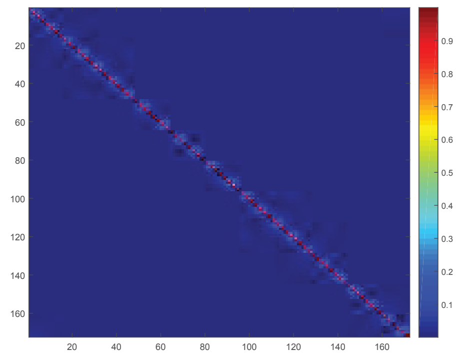

Figure 6: Multiplicative uncertainty in left harmonic matrix, $\hat{\Phi}_0$ ($\Delta_{\hat{\Phi}}$).

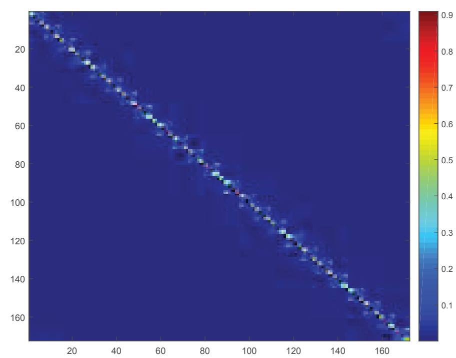

Figure 7: Multiplicative uncertainty in right harmonic matrix, $\hat{\Psi}_0$ ($\Delta_{\hat{\Psi}}$).

affects the Fourier coefficients, represented by $\Delta_{\hat{\Sigma}}$ shown in Fig. 8. The uncertainty is diagonally structured and the largest error is seen at the modes associated with the tune i.e. the 13th mode (where $\nu_y = 13.36$). For this calculation $F = 40$ was chosen. However, given the relationship in Eq. 24, the uncertainty in the tune can be expressed as

$$\Delta_\nu = W_{\hat{\Sigma}}^{-1} \Delta_{\hat{\Sigma}} \tag{27}$$

and using this relationship, size of uncertainty in the tune at each harmonic harmonic modes is shown in Fig. 8. Therefore for a given error in each Fourier coefficient, the resulting change in tune at each harmonic mode can be determined.

CONCLUSION

Uncertainty in the response matrix can be described by multiplicative uncertainty descriptions using either a Singular Value or Harmonic decompositions of the response matrix. Using the SVD approach, the uncertainty in the

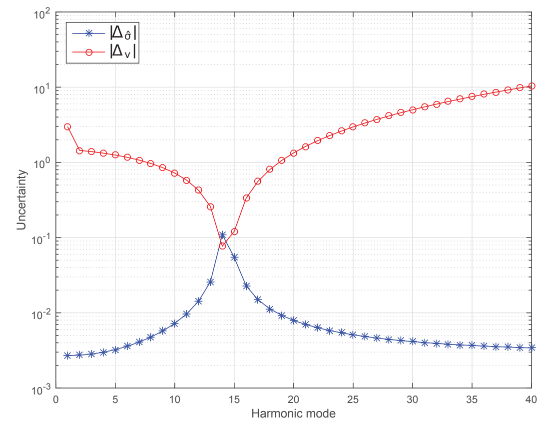

Figure 8: Multiplicative uncertainty in Fourier coefficients, $\hat{\Sigma}_0$ ($\Delta_{\hat{\Sigma}}$) and in the tune across modes (Δ_ν).

response matrix can be expressed in terms of the left and right singular vectors and the singular values. This is useful for assessing the stability of the closed loop which is based on SVD, however it is difficult to interpret the uncertainty description in terms of physical beam parameters and to decouple the error observed at a singular BPM or corrector. The Fourier approach on the other hand, offers uncertainty descriptions which are directly associated with the betatron function and phase advance changes at BPM and corrector locations. Furthermore, errors in the Fourier coefficients are directly associated with tune drifts. Therefore the Fourier approach offers physical interpretation of the uncertainty observed in the response matrix. In the companion paper in these proceedings, the uncertainty descriptions are used to determine closed loop stability [1].

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