A New Paradigm for Modeling, Simulation, and Analysis of Intense Beams

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Overview

• Purpose is to create a transfer map which includes the effects of space charge.
• It is done self consistently.
• It can be used for fitting and optimization.
• Includes a fast multipole method solver for high accuracy without averaging.
Outline

• Overview of differential algebras and numerical derivatives
• Overview of normal form methods
• Calculation of space charge maps and their inclusion into the element.
• Example systems
• Further expansions
• Conclusions
Differential Algebras and Numerical Derivatives

First we introduce the $D_1$ Differential Algebra, which is a first order one variable vector.

\[
(q_0, q_1) + (r_0, r_1) = (q_0 + r_0, q_1 + r_1)
\]

\[
n(q_0, q_1) = (nq_0, nq_1)
\]

\[
(q_0, q_1)(r_0, r_1) = (q_0r_0, q_0r_1 + r_0q_1)
\]

It follows that,

\[
(f(x), f'(x)) = f(x + d)
\]

\[
d = (0, 1)
\]

As an example,

\[
f(x) = 2x^2 - x + 3
\]

\[
f'(x) = 4x - 1
\]

\[
f(2) = 9
\]

\[
f'(2) = 7
\]

\[
f(2 + d) = f(2, 1)
\]

\[
= 2(2, 1)(2, 1) - (2, 1) + (3, 0)
\]

\[
= 2(4, 4) - (2, 1) + (3, 0)
\]

\[
= (8, 8) - (2, 1) + (3, 0)
\]

\[
= (9, 7)
\]
Normal Form Methods

\[ N = \mathcal{A} \circ \mathcal{M} \circ \mathcal{A}^{-1} \]
Adding in Space Charge

• We need to solve Poisson’s Equation

• Fast

• Accurate

• Valid throughout the distribution.
The Distribution Function

\[ \rho(x, y) = \sum_i \delta(x - x_i) \delta(y - y_i) \rightarrow \rho(x, y) = \sum_j \sum_k C_{jk} x^j y^k \]

- Mathematically if two distributions have the same moments then they are mathematically indistinguishable.
\[ M_{ij} = \int \int x^i y^j \sum_l \sum_m C_{lm} x^l y^m \, dx \, dy \]

\[ M = TC \]

\[ C = T^{-1} M \]
Potential Calculation

- Since we now have a Taylor series for the distribution can’t we just integrate and find the potential.

- No, Since the expansion is occurring inside the distribution Singularities become an issue.

\[ G(r, r') = \ln(|r - r'|); \lim_{r \to r'} (G(r, r')) = \infty \]
Duffy Transformation
Duffy Transformation

\[ \int\int \ln\left(\sqrt{(x-x_0)^2 + (y-y_0)^2}\right) \, dx \, dy \]
Duffy Transformation

\[ \int_{y_0}^{y_1} \int_{x_0}^{x_1} \ln(\sqrt{(x-x_0)^2 + (y-y_0)^2}) \, dx \, dy + \int_{y_0}^{y_2} \int_{x_0}^{x_2} \ln(\sqrt{(x-x_0)^2 + (y-y_0)^2}) \, dx \, dy + \int_{y_0}^{y_3} \int_{x_0}^{x_3} \ln(\sqrt{(x-x_0)^2 + (y-y_0)^2}) \, dx \, dy + \int_{y_0}^{y_4} \int_{x_0}^{x_4} \ln(\sqrt{(x-x_0)^2 + (y-y_0)^2}) \, dx \, dy \]
Duffy Transformation

\[ \int_{y_0}^{y_d} \int_{x_0}^{x_b} \ln\left( \sqrt{(x - x_0)^2 + (y - y_0)^2} \right) dx \, dy \]
Duffy Transformation

\[ u_1 = \frac{x - x_0}{b - x_0}; \quad u_2 = \frac{y - y_0}{d - y_0}; \quad \lambda_1 = (b - x_0); \quad \lambda_2 = (d - y_0); \]

\[
\lambda_1 \lambda_2 \iint_{0,0}^{1,1} \ln(\sqrt{(\lambda_1 u_1)^2 + (\lambda_2 u_2)^2}) \, dx \, dy
\]
Duffy Transformation

\[
\begin{array}{|c|c|}
\hline
0 & 1 \\
\hline
\end{array}
\]
Duffy Transformation
Duffy Transformation

![Duffy Transformation Graph](image)
Duffy Transformation
Duffy Transformation
Duffy Transformation

• This is done using the following set of coordinate transformations:

\[ w_1 = u_1; w_2 = \frac{u_2}{u_1} \]

\[ \lambda_1 \lambda_2 \int_0^1 \int_0^1 w_1 \ln(\sqrt{\lambda_1^2 w_1^2 + \lambda_2^2 w_1^2 w_2^2}) dw_1 dw_2 \]
Duffy Transformation
Duffy Transformation
Duffy Transformation
Duffy Transformation
Duffy Transformation
Duffy Transformation

- This is done using the similar set of coordinate transformations:

\[ w_2 = u_2; \quad w_1 = \frac{u_1}{u_2} \]

\[ \lambda_1 \lambda_2 \int_0^1 \int_0^1 w_2 \ln(\sqrt{\lambda_2^2 w_2^2 + \lambda_1^2 w_2^2 w_1^2}) \, dw_1 \, dw_2 \]
Duffy Transformation

\[ \lambda_1 \lambda_2 \int_0^1 \int_0^1 w_1 \ln(w_1) \, dw_1 \, dw_2 + \ln(\sqrt{\lambda_1^2 + \lambda_2^2 w_2^2}) \, dw_1 \, dw_2 \]

\[ \lambda_1 \lambda_2 \int_0^1 \int_0^1 w_2 \ln(w_2) \, dw_1 \, dw_2 + \ln(\sqrt{\lambda_1^2 w_1^2 + \lambda_2^2}) \, dw_1 \, dw_2 \]

\[ \lim_{x \to 0} (x \ln(x)) = 0 \]
External Fields

- These are modeled using Strang Splitting.

\[
\begin{align*}
\text{Diffeq1}(L) & \rightarrow \text{Solution1}(L) \\
\text{Diffeq2}(L) & \rightarrow \text{Solution2}(L) \\
\text{Diffeq1}(L) + \text{Diffeq2}(L) & \rightarrow ?
\end{align*}
\]
External Fields (cont’d)

\[ \mathcal{M}(L/2) \quad \text{Kick}(L) \quad \mathcal{M}(L/2) + O(L^3) \]
Accuracy for Various Orders and Moments

Accuracy
Example

\[ \frac{r_m}{r_0} = 1 + 5.87 \times 10^{-5} \frac{I}{(\gamma^2 - 1)^{\frac{3}{2}}} \left( \frac{z}{r_0} \right)^2 \]

<table>
<thead>
<tr>
<th>Method</th>
<th>Growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>Edge Point x</td>
<td>35.27 %</td>
</tr>
<tr>
<td>Edge Point y</td>
<td>35.30 %</td>
</tr>
<tr>
<td>Map Element x</td>
<td>31.21 %</td>
</tr>
<tr>
<td>Map Element y</td>
<td>31.34 %</td>
</tr>
</tbody>
</table>
Tune Measurement

Fractional X Tune

Quad Current

Fractional Y Tune

Fractional Tune
Chromaticity

![Chromaticity Graph]

Fractional X Tune

Chromaticity

![Chromaticity Graph]
Limitations
Fast Multipole Method

• Uses multipole expansions for distant particles
• Uses direct coulomb interactions for close particles
• Currently used in solid state physics, fluids, and chemistry
FMM: Examples
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FMM: Examples
\[
\phi(z) = \sum_i \left( Q \log(z_i) + \sum_k \frac{a_k}{z_i^k} \right)
\]
\[ \phi(z) = \sum_i \left( Q \log(z_i) + \sum_k \frac{a_k}{z_i^k} \right) \]
$\phi(z) = \sum_i \left( Q \log(z_i) + \sum_k \frac{a_k}{z_i^k} \right)$
FMM: Examples

\[ \phi(z) = \sum_i \left( Q \log(z_i) + \sum_k \frac{a_k}{z_i^k} \right) \]
\[ \phi(z) = \sum_i \left( Q \log(z_i) + \sum_k \frac{a_k}{z_i^k} \right) + \sum_j q_j \log(z_j) \]
Conclusions

• Adding the effects of space charge to the transfer map of a system is both possible and feasible
• New insights can be found using this method in conjunction with normal form analysis
• Can even analyze complex distributions using the fast multipole method
Questions?
Map Overview

- A map is a method of advancing particles which takes the form

\[
\begin{align*}
    z_f &= M z_i \\
    
    \begin{pmatrix}
        x_f \\
        p_{xf}
    \end{pmatrix} &= \begin{pmatrix}
        (x_f | x_i) & (x_f | p_{xi}) \\
        (p_{xf} | x_i) & (p_{xf} | p_{xi})
    \end{pmatrix} \cdot \begin{pmatrix}
        x_i \\
        p_{xi}
    \end{pmatrix}
\end{align*}
\]
A map is a method of advancing particles which takes the form

$$z_f = Mz_i$$

$$\begin{align*}
\dot{x}_f &= (x_f | x_i)x_i + (x_f | p_{xi})p_{xi} x_i \\
\dot{p}_{xf} &= (p_{xf} | x_i)x_i + (p_{xf} | p_{xi})p_{xi} \dot{x}_i
\end{align*}$$
A map is a method of advancing particles which takes the form

\[ z_f = M z_i \]

\[
x_f = (x_f | x_i) x_i + (x_f | p_{xi}) p_{xi} + (x_f | x_i^2) x_i^2 + (x_f | x_i p_{xi}) x_i p_{xi} + (x_f | p_{xi}^2) p_{xi}^2
\]

\[
p_{xf} = (p_{xf} | x_i) x_i + (p_{xf} | p_{xi}) p_{xi} + (p_{xf} | x_i^2) x_i^2 + (p_{xf} | x_i p_{xi}) x_i p_{xi} + (p_{xf} | p_{xi}^2) p_{xi}^2
\]
Legendre Polynomials

• Legendre Polynomials have the following orthogonality property.

\[ \int_{-1}^{1} P_m(x) P_n(x) dx = \frac{2}{2n+1} \delta_{mn} \]

• If we assume that the distribution can be modeled as a sum of Legendre polynomials, we can easily find the coefficients.

\[ \rho(x) = \sum_{n} C_n P_n(x) \]

\[ \int_{-1}^{1} P_m(x) \rho(x) dx = C_m \frac{2}{2m+1} \]
Method Comparison

13th Order Legendre

13th Order Moment Method

21st Order Legendre

21st Order Moment Method
Particle Advancement
Particle Advancement Cont’d