The effect of undulator harmonics field on Free-Electron Laser harmonic generation

Qi-ka, Jia

National Synchrotron Radiation Laboratory
University of Science and Technology of China
Hefei, Anhui 230029, China
Jiaqk@ustc.edu.cn
Outline

• Introduction
• Analysis
  in an ideal undulator
  in an actual undulator
  3\textsuperscript{rd} harmonic case
• Summary
• Using the higher harmonic:
  
a way for FEL => shorter $\lambda$s.

$$\lambda_{sn} = \frac{\lambda_u}{2n\gamma^2} (1 + a_u^2)$$

• For a planar undulator with $B_u \sin(k_u z)$,
  the electron's non-uniform axial motion ($\beta_z$)
  => the odd harmonics radiations on axis
  $n=1,3,5,\ldots$. 
The harmonic radiation can be enhanced by $\uparrow B_{un}$.

Some methods for this were proposed, *eg.*:

- Putting high permeability shims inside the undulator **
- Optimizing magnetic blocks size in a standard Harbch undulator ***

Here, we analyze the effect of $B_n$ on FEL harmonic generation.

---

*** Qi-ka Jia, “Undulator Harmonic field enhancement analysis”, Proceedings of IPAC10, WEPD033/3165
In a ideal planar undulator the e-s oscillate*

at odd harmonics in the transverse direction

\[ \beta_x \approx -4 \sum_{n=1,3,5,...} \left( \frac{K}{4\gamma} \right)^n \frac{(-1)^{(n-1)/2}}{[(n-1)/2]!} \cos[nk_u \bar{z}] \]

at even harmonics in the axial direction

\[ \beta_{II} \approx \bar{\beta}_{II} + 2 \sum_{n=2,4,...} \left( \frac{K}{4\gamma} \right)^n \frac{(-2)^n}{[(n-2)/2]!} \cos[nk_u \bar{z}] \]

=> radiations

on-axis odd harmonics

even harmonics off-axis

*Qika Jia, “Harmonic motion of electron trajectory in planar undulator,” PAC09-WE5RFP088
For FEL

the \( n \)th harmonic optical field equation
and the phase equation in 1-D mode:

\[
\frac{d}{dz} \tilde{a}_{sn} = \frac{r_e n_e a_u [J, J]_n \lambda_{sn}}{\gamma} \langle e^{-in\phi} \rangle
\]

\[
\frac{d^2 \phi}{dt^2} = -\frac{c^2}{\gamma^2} 2a_u k_u \text{Re} \sum_n [J, J]_n k_{sn} \tilde{a}_{sn} e^{in\phi}
\]

the coupling coefficient:

\[
[J, J]_n = (-1)^{n-1} \frac{na_u^2}{2(1 + a_u^2)} \left( J_{\frac{n-1}{2}}^2 - J_{\frac{n+1}{2}}^2 \right)
\]

\( \phi = (k_s + k_u)z - \omega_s t \)
the harmonic generation can be characterized by the coupling coefficients*

- Small signal gain in low gain FEL

\[
g_n = -n \left( \frac{[J,J]_n}{[J,J]_1} \right)^2 (4\pi N \rho)^3 \left\langle \frac{\partial}{\partial x} \sin c^2 \frac{x}{2} \right\rangle_{\phi_0},
\]

- Nonlinear harmonic generation in high gain FEL

\[
\frac{P_n}{\rho P_e} \left( \frac{n^{n-1}[J,J]_n}{n! [J,J]_1} \right)^2 \left( \frac{P_{10}}{9 \rho P_e} \right)^n e^{-\frac{n^2}{L_g}}
\]

harmonic saturation power: \[
\frac{P_{ns}}{P_{1s}} \approx \frac{(n+1)^n}{2(n*n!)^2} \left( \frac{[J,J]_n}{[J,J]_1} \right)^2
\]

the harmonic coupling coefficient
with undulator deflection parameter

harmonic generation $\propto \left( [J, J]_n / [J, J]_1 \right)^2$
For actual planar undulators

\[ B_u = \sum_m B_{um} \sin(mk_u z) \quad \tilde{a}_u = \sum_m \hat{a}_{um} \cos(mk_u z) \]

\( m: \) all or part odd numbers, due to the symmetry of the magnetic structure

generally \( B_{um} \ll B_{u1}, \quad a_{um} = \frac{B_{um}}{mB_{u1}} a_{u1} \ll a_{u1} \)

• Resonance condition

\[ \lambda_{sn} = \frac{\lambda_u}{2n\gamma^2}(1 + \sum_m a_{um}^2) \]

\( a_{um}^2 = \hat{a}_{um}^2 / 2, \) (rms)
**Electron motion**

Using relations

\[ \beta_{\perp}^2 = \frac{\tilde{a}_u^2}{\gamma^2}, \quad \beta_u = 1 - \frac{1}{2} \left( \frac{1}{\gamma^2} + \beta_{\perp}^2 \right) \]

give

\[ z = \bar{z} - \left\{ \sum_m \frac{\xi_m}{k_s} \sin(2k_u \bar{z}) + \sum_{m \neq l} \frac{\xi_{ml+}}{k_s} \sin((m+l)k_u \bar{z}) + \frac{\xi_{ml-}}{k_s} \sin((m-l)k_u \bar{z}) \right\} \]

where

\[ \xi_m = \frac{a_{um}^2}{2m(1+\sum a_{ui}^2)} = \frac{r_m^2}{m} \xi_1, \quad \xi_{ml\pm} = \frac{a_{um}a_{ul}}{(m\pm l)(1+\sum a_{ui}^2)} = \frac{2r_mr_l}{m\pm l} \xi_1 \]

because

\[ r_m = \frac{a_{um}}{a_{u1}} = \frac{B_{um}}{mB_{u1}} << 1, \quad \xi = \frac{a_{u1}^2}{2(1+\sum a_{ui}^2)} < \frac{a_{u1}^2}{2(1+a_{u1}^2)} < \frac{1}{2} \]

\[ \xi_m, \quad \xi_{ml\pm} << 1, \text{ for } m, l \neq 1 \]

Only the terms related with the fundamental are dominant therefore

\[ z \square \bar{z} - \left\{ \frac{\xi_1}{k_s} \sin(2k_u \bar{z}) + \sum_{m \neq 1} \frac{\xi_{m+}}{k_s} \sin((m+1)k_u \bar{z}) + \frac{\xi_{m-}}{k_s} \sin((m-1)k_u \bar{z}) \right\} \]
the phase equation

\[
\phi'' = \frac{2k_u}{\gamma^2} \sum_{n,l} k_{sn} a_{sn} a_{ul} \{ \cos[(nk_u + lk_u) z - n\omega t + \phi] + \cos[(nk_u - lk_u) z - n\omega t + \phi] \}
\]

substituting expression of \( z \)

\[
\phi'' = \frac{2k_u}{\gamma^2} \sum_n k_{sn} a_{sn} a_{u1} f_n \text{Re} e^{-i(n\phi + \phi_{sn})}
\]

\[
f_n = \text{Re} \sum_l \frac{a_{ul}}{a_{u1}} [e^{i(n-l)k_u \xi} + e^{i(n+l)k_u \xi}] e^{i\xi \sin(2k_u \xi) \sum_{m \neq 1} e^{i\xi \sin((m-1)k_u \xi) z}}
\]

In the exponential of \( f_n \), many terms are small and oscillate fast, an average over undulator period will eliminate these small contribution terms.

the \( n \)th harmonic optical field equation

\[
\frac{d}{dz} \tilde{a}_{sn} \int \frac{r_{ne} a_{u1} \lambda_{sn}}{\gamma} f_n \langle e^{-in\phi} \rangle
\]

the modified coupling coefficient: \([J, J]_n \rightarrow f_n\)
Because it can be further simplified by taking $h_2 = 0$:

$$B_u = B_{u1} \sin(k_u z) + B_{u3} \sin(3k_u z)$$

in this case:

$$z = \bar{z} - \frac{\zeta_1}{k_{s1}} \sin(2k_u \bar{z}) - \frac{\zeta_2}{k_{s1}} \sin(4k_u \bar{z})$$

$$\xi_1 = \frac{a_{u1}(a_{u1} + a_{u3})}{2(1 + a_{u1}^2 + a_{u3}^2)} , \quad \xi_2 = \frac{a_{u1}a_{u3}}{4(1 + a_{u1}^2 + a_{u3}^2)}$$

We have

$$f_n = \text{Re} \sum_l \frac{a_{ul}}{a_{u1}} \left[ e^{i(n-l)k_u \bar{z}} + e^{i(n+l)k_u \bar{z}} \right] \sum_{h_1} \sum_{h_2} J_{h_1}(n\zeta_1)J_{h_2}(n\zeta_2) e^{i(h_1 + 2h_2)2k_u \bar{z}}$$

Taking average over undulator period

the dominant product term in the sum is that with $h_1 + 2h_2 = -(n \pm l) / 2$

$$f_n = \sum_l \frac{a_{ul}}{a_{u1}} \left\{ \sum_{h_1, h_2, h_1 + 2h_2 = -\frac{n+l}{2}} J_{h_1}(n\zeta_1)J_{h_2}(n\zeta_2) + \sum_{h_1, h_2, h_1 + 2h_2 = -\frac{n-l}{2}} J_{h_1}(n\zeta_1)J_{h_2}(n\zeta_2) \right\}$$

for the small arguments, only zero order Bessel function contribute

Because $\zeta_2 \ll \zeta_1 < 1/2$, it can be further simplified by taking $h_2 = 0$:
the modified coupling coefficient:

\[
[J, J]_1 \rightarrow f_1 = J_0(\zeta_2)\left\{J_0(\zeta_1) - J_1(\zeta_1)\right\} + \frac{a_{u3}}{a_{u1}} \left[J_2(\zeta_1) + J_1(\zeta_1)\right] \\
[J, J]_3 \rightarrow f_3 = J_0(3\zeta_2)\left\{J_2(3\zeta_1) - J_1(3\zeta_1)\right\} + \frac{a_{u3}}{a_{u1}} \left[J_0(3\zeta_1) - J_3(3\zeta_1)\right]
\]

3rd harmonic generation \( \propto (f_3 / f_1)^2 \)

enhancement of the 3rd harmonic

\[
R_3 = \left(\frac{f_3 / f_1}{[J, J]_3 / [J, J]_1}\right)^2
\]
numerical calculation result

Modified coupling coefficient due to 3\textsuperscript{rd} harmonic magnetic field
3\textsuperscript{rd} harmonic generation $\propto (f_3/f_1)^2$

the effect of $B_3$ on FEL harmonic generation
The enhancement of the FEL 3rd harmonic radiation

\[ R_3 = \left( \frac{f_3 / f_1}{[J, J]_3 / [J, J]_1} \right)^2 \]

argument: \( \frac{a_{u1}}{a_u} \) has a little difference:

\[ a_u^2 = a_{u1}^2 [1 + \left( \frac{a_{u3}}{a_{u1}} \right)^2] \Rightarrow \text{same resonant wavelength} \]
- 3rd harmonic emission can be distinctly enhanced by 3rd harmonic field with an opposite sign to fundament field
- the larger magnetic harmonics fractions
  => the larger radiation enhancement,
  \[ \left| \frac{B_{u3}}{B_{u1}} \right| \uparrow \Rightarrow a_{s3} \uparrow \]
- fundamental radiation has been less affected
- for given \( \frac{B_{u3}}{B_{u1}} \)
  the weaker \( a_u \) => larger enhancement of \( a_{s3} \)

With \( \frac{B_{u3}}{B_{u1}} = -0.3 \):
  the 3rd-harmonic radiation are enhanced \( \sim 40\% \)
  maximally doubled for \( K(=\sqrt{2a_u}) \sim 1 \)
SUMMARY

- Effects of undulator harmonics field on the harmonic coupling coefficients and FEL harmonic generation are analysed.
- For the case 3rd magnetic field present, analytical expression is given for coupling coefficients, is easy to calculate and can be used to predict the enhancement of FEL HG.
- 3rd emission increase with 3rd magnetic field that has an opposite sign to $B_1$.
- Fundamental emission has been less affected.
- Next work: further study by simulation.
ACKNOWLEDGE

Work supported by the National Nature Science Foundation of China under Grant No. 10975137

REFERENCES

[8] Qi-ka Jia, PAC09-WE5RFP088
Thank you