THE PHYSICS OF FEL IN AN INFINITE ELECTRON BEAM*

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Abstract
We solve linearized Vlasov-Maxwell FEL equations for a 3-D perturbation in an infinite electron beam with Lorentzian energy distributions using paraxial approximation. We present analytical solutions for various initial perturbations and discuss the effect of optical guiding in such system.

INTRODUCTION
Developing complete theoretical model of Coherent electron Cooling (CeC) [1] is important for gaining insights into the physics of the processes, studying the scaling law and benchmarking simulation codes. Deriving analytical formula under certain assumptions is one of the key-stone in this process. For instance, the modulation process can be described by a close form solution obtained for an infinite electron beam with kappa-2 velocity distribution [2]. This solution is applicable to a realistic case when the transverse Debye radii are much smaller than the transverse size of electron beam.

In this work, we try to derive an analytical 3-D solution for the FEL amplification process under assumption of infinitely wide electron beam. 1D FEL theories has been applied to the amplification process in CeC, naturally assuming an infinite electron beam and longitudinally propagating radiation fields, i.e. $k_\perp = 0$ [3]. While 1D FEL theory provides closed-form analytical solutions for certain energy distributions, the diffraction effects are ignored. Hence, the transverse profile of the amplified modulation can not be obtained. In present day analytical 3D FEL theory, applied to specific spatial profiles of electron beam [4-6], the solutions are usually expanded into infinite number of modes determined by specific boundary conditions. In the high gain limit, the transverse profile of the electron modulation is determined by the mode with largest growth rate. However, for FEL with nominal or relatively short length, transient effect may not be ignored and thus presents difficulties in analytical evaluation of the amplification.

In order to incorporate the diffraction effects into analytical solution capable of describing the transient effect, we investigate the FEL amplification process for an infinite electron beam. The results derived under this assumption are applicable if the electron beam size is much larger than that of the amplified current modulation. Similarly to 1D FEL model, we assume the unperturbed electron spatial density is a constant and electrons are moving along their trajectory determined by the undulator field with no transverse dynamic effects from the radiation field or space charge. However, we allow the radiation to propagate with an angle with respect to the longitudinal direction, i.e. $k_\perp \neq 0$. Starting from the self-consistent paraxial field equation, we arrive to a third order ordinary differential equation (ODE). Analytical solutions are obtained for various initial conditions and the effect of optical guiding is discussed.

EQUATION OF MOTION
We use standard assumptions that the amplitude of the radiation field varies slowly with respect to the undulator period and that fast oscillation terms can be dropped. The paraxial equation on the amplitude of the radiation field is [4]

$$
\begin{align*}
\mathbf{\nabla}^2 \tilde{E}(z,r_\perp,C) + 2i \frac{\omega}{c} \frac{\partial}{\partial z} \tilde{E}(z,r_\perp,C) \\
= \int_0^\pi \int_0^\pi j_0(r_\perp) d\theta' d\phi' \frac{2\pi}{c^2} \mathbf{\mathbf{\gamma}}^2 \omega_0 \tilde{E}(z',r_\perp,C) \\
+ \frac{4\pi \omega}{\omega_0} \int_0^\pi \int_0^\pi \int_0^\pi \mathbf{\nabla}^2 \tilde{E}(z',r_\perp,C) d\theta' d\phi' d\theta \\
\times \int dP \frac{dF}{dP} \exp \left[ i \left( C + \frac{\omega_0 P}{\gamma_0} \right) \left( z'-z \right) \right]
\end{align*}
$$

(1)

where $\tilde{E}(z,r_\perp,C)$ is the complex amplitude of the radiation field, $\omega$ is the radiation frequency, $C$ is the detuning, $\mathbf{\gamma}_0$ is the nominal electron energy, $P$ is the electron energy deviation, $\theta$ is the electron deflection angle, $F(P)$ is the energy distribution function and $j_0(r_\perp)$ is the transverse spatial distribution of the unperturbed electron beam. Assuming $j_0(r_\perp) = \delta_0 \mathbf{\gamma}$, the Fourier transformation of eq. (1) with respect to transverse spatial coordinates $x$ and $y$ yields

$$
\begin{align*}
\frac{k_\perp^2 \mathbf{\mathbf{\gamma}}}{2\omega} \tilde{E}(z,k_\perp,C) + \frac{\partial}{\partial z} \tilde{E}(z,k_\perp,C) \\
= \frac{\pi j_0 \mathbf{\mathbf{\gamma}}^2}{c} \int dz' \left[ \tilde{E}(z',k_\perp,C) + \frac{4i c}{\mathbf{\mathbf{\gamma}}^2 \omega_0} \frac{k_\perp^2 \mathbf{\mathbf{\gamma}}}{2\omega} \tilde{E}(z',k_\perp,C) \\
+ \frac{\partial}{\partial z} \tilde{E}(z',k_\perp,C) \right] dP \frac{dF}{dP} \exp \left[ i \left( C + \frac{\omega_0 P}{\gamma_0} \right) \left( z'-z \right) \right]
\end{align*}
$$

(2)

Inserting the definition

$$
\tilde{R}(z,k_\perp,C) \equiv \mathbf{\mathbf{\gamma}} \mathbf{\mathbf{\gamma}}^2 \tilde{E}(z,k_\perp,C),
$$

(3)

into eq. (2), we get the following:

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\[
\frac{\partial}{\partial z} \tilde{R}(z, k_x, C) = \frac{m_i e^2}{c} \int_0^\infty dz' e^{ik_x(z'-z)} \left\{ \tilde{R}(z', k_x, C) + \frac{4ic}{\theta^2} \frac{\partial}{\partial z} \tilde{R}(z', k_x, C) \right\}
\]

Using normalized variables defined in [4] and [7], eq. (4) becomes
\[
\frac{\partial}{\partial \tilde{z}} \tilde{R}(\tilde{z}, \tilde{k}_x, \tilde{C}) = \frac{1}{\Gamma} \int_0^\infty d\tilde{z}' e^{i(k_x \tilde{z}' - \tilde{z}')} \left\{ \tilde{R}(\tilde{z}', \tilde{k}_x, \tilde{C}) + i\Lambda_{\rho} \frac{\partial}{\partial \tilde{z}} \tilde{R}(\tilde{z}', \tilde{k}_x, \tilde{C}) \right\},
\]

where \( \tilde{z} = z / \Gamma \), \( \tilde{C} \equiv C / \Gamma \), \( \Gamma \) is the gain parameter:
\[
\Gamma \equiv \left[ \frac{4\pi n_i \theta^2}{\gamma^2 \hbar \omega} \right]^{1/2},
\]
and \( \tilde{\rho}(\tilde{p}) \) is the energy distribution function satisfying
\[
\int \tilde{\rho}(\tilde{p}) = 1.
\]
In order to proceed further, we assume Lorentzian energy distribution, i.e.
\[
\tilde{\rho}(\tilde{p}) = \left( 1 + \frac{p^2}{\tilde{q}^2} \right)^{-1/2},
\]
Inserting eq. (6) into eq. (5) results in the following:
\[
\frac{\partial}{\partial \tilde{z}} \tilde{R}(\tilde{z}, \tilde{k}_x, \tilde{C}) = -i \int_0^\infty d\tilde{z}' (\tilde{z}' - \tilde{z}) e^{i(k_x \tilde{z}' - \tilde{z}')} \left\{ \tilde{R}(\tilde{z}', \tilde{k}_x, \tilde{C}) - i\Lambda_{\rho} \frac{\partial}{\partial \tilde{z}} \tilde{R}(\tilde{z}', \tilde{k}_x, \tilde{C}) \right\},
\]

It has been demonstrated that integro-differential equation with the form of (7) can be reduced to a third-order ODE (see Chapter 6.3.3 and 6.3.4 of [7]). Eq. (7) is transformed into
\[
\frac{d^3}{d\tilde{z}^3} \tilde{R}(\tilde{z}) + 2i(\tilde{C}_{\lambda} + \tilde{q}) \frac{d^2}{d\tilde{z}^2} \tilde{R}(\tilde{z}) + \left[ \Lambda_{\lambda} + i(\tilde{C}_{\lambda} + \tilde{q}) \right] \frac{d}{d\tilde{z}} \tilde{R}(\tilde{z}) - i\tilde{R}(\tilde{z}) = 0,
\]
where we defined a new variable as
\[
\tilde{C}_{\lambda} \equiv \tilde{C}_\lambda - \tilde{k}_\lambda^2.
\]
The solution of (8) is the sum of three eigen-modes, i.e.
\[
\tilde{R}(\tilde{z}) = \sum_{j=1}^3 A_j(\tilde{C}, \tilde{k}_\lambda) e^{\lambda_j \tilde{z}},
\]
where the eigen-value \( \lambda_j \) are determined by
\[
\lambda_j^3 + 2i(\tilde{C}_{\lambda} + \tilde{q}) \lambda_j^2 + \left[ \Lambda_{\lambda} + i(\tilde{C}_{\lambda} + \tilde{q}) \right] \lambda_j - i = 0
\]
and \( A_j(\tilde{C}, \tilde{k}_\lambda) \) are determined by initial conditions at the FEL entrance. From eq. (3) and eq. (10), the complex amplitude of the radiation field is given by
\[
\tilde{E}(\tilde{z}, \tilde{k}_x, \tilde{C}) = e^{-i\tilde{q} \tilde{z}} \sum_{j=1}^3 A_j(\tilde{C}, \tilde{k}_\lambda) e^{\lambda_j \tilde{z}}.
\]

\[\text{Excitation with External Field}\]

If the initial seeding of FEL is solely from external field, the coefficients \( A_j \) can be derived from eq. (16) as
\[
A_j(\tilde{C}, \tilde{k}_\lambda) = \frac{E_{\text{ext}}(\tilde{C}, \tilde{k}_\lambda)}{\Lambda_{\lambda} - \lambda_j \lambda_j + \lambda_j \lambda_3 + \lambda_3 \lambda_1},
\]
where \( i, j, k = 1, 2, 3 \) and \( \varepsilon_{ijk} \) is the Levi-Civita symbol. For simplicity, we assume the transverse profile of the external field is Gaussian:
\[
\tilde{E}_{\text{ext}}(\tilde{C}, \tilde{k}_\lambda) = \tilde{E}_{\text{ext}} F_{\text{ext}}(\tilde{C}) \exp(\tilde{k}^2 \Sigma_{\text{ext}}),
\]
where \( \Sigma_{\text{ext}} \) is a parameter describing the transverse range of the external field, \( F_{\text{ext}}(\tilde{C}) \) is a function describing the frequency content of the external field and \( \tilde{E}_{\text{ext}} \) is a parameter determining the strength of the excitation.
**Instantaneous Pulse**

For an instantaneous excitation at the FEL entrance $\tilde{z} = 0$ described by Dirac delta-function $\delta(t)$, $F_w(\tilde{C})$ is just a constant independent of $\tilde{C}$. Without loss of generality, we assume

\[ F_w(\tilde{C}) = 1. \]  

Thus the external field is written as

\[ \tilde{E}_{\text{ext}}(\tilde{k}_x, \tilde{k}_y) = \tilde{E}_{\text{in}} \exp\left(-\tilde{k}_y^2 \tilde{\sigma}_y^2 \right), \]  

and the coefficients $A_j$ is

\[ A_j(\tilde{C}_{3d}, \tilde{k}_x) = \tilde{E}_{\text{in}} A_{m,j}(\tilde{C}_3) \exp\left(-\tilde{k}_y^2 \tilde{\sigma}_y^2 \right) \]  

with

\[ A_{m,j}(\tilde{C}_3) = \frac{e_{\text{in}} \lambda_j \lambda_i}{\tilde{\lambda}_i^2 - \lambda_j \lambda_i + \lambda_j \lambda_i}. \]  

Inserting eq. (21) into eq. (15) generates

\[ j_1(\tilde{x}, \tilde{t}) = -\frac{c^2 \tilde{E}_{\text{in}} e^{-i2\pi |\tilde{r}|^2}}{2\pi \tilde{\omega}} \times \sum_{i=1}^{3} A_i(\tilde{C}_3) \tilde{\lambda}_i(\tilde{C}_3) e^{i|\tilde{\lambda}_i|^2 |\tilde{r}|^2} d\tilde{C}_3 \]  

The current density modulation is given by the inverse Fourier transformation of (23) with respect to $k_x$ and $k_y$ and $\tilde{C}_3$, i.e.

\[ \tilde{j}_1(\tilde{x}, \tilde{t}) = \frac{c^2 \Gamma^2 \sigma_t^2 \tilde{E}_{\text{in}}}{8\pi^2 \tilde{\theta}^3} e^{i2\tilde{\omega}\tilde{t}} e^{i2\tilde{\omega}|\tilde{r}|^2} \int dk_x e^{i\tilde{k}_x k_x} \int dk_y e^{i\tilde{k}_y k_y} \]

\[ \times e^{-i2\tilde{\omega}|\tilde{r}|^2} \sum_{i=1}^{3} A_i(\tilde{C}_3) \tilde{\lambda}_i(\tilde{C}_3) e^{i|\tilde{\lambda}_i|^2 |\tilde{r}|^2} d\tilde{C}_3 \]  

Since the Jacobian

\[ J = \left| \frac{\partial(\tilde{C}_3, \tilde{k}_x, \tilde{k}_y)}{\partial(\tilde{C}_3, \tilde{k}_x, \tilde{k}_y)} \right| = 1, \]

changing in eq. (24) the integration variable from $\tilde{C}_3$ to $\tilde{C}_{3d}$ and integrating over $k_x$ and $k_y$ gives:

\[ j_1(\tilde{x}, \tilde{t}) = \frac{c^2 \Gamma^2 \sigma_t^2 \tilde{E}_{\text{in}}}{4\pi \tilde{\omega}^3} e^{i2\tilde{\omega}\tilde{t}} e^{i2\tilde{\omega}|\tilde{r}|^2} \frac{|\tilde{\lambda}_i|^2}{|\tilde{\lambda}_i|^2 - |\tilde{\lambda}_i|^2} \]

\[ \times \sum_{i=1}^{3} A_i(\tilde{C}_{3d}) \tilde{\lambda}_i(\tilde{C}_{3d}) e^{i|\tilde{\lambda}_i|^2 |\tilde{r}|^2} d\tilde{C}_{3d} \]  

where

\[ \tilde{x}(z, t) = \Gamma^{-1} \left( 2|\tilde{r}|^2 |z - ct| + z \right). \]  

As seen from eq. (26), the longitudinal evolution of the current density is identical to that in the 1D FEL theory, while the transverse evolution is described by a Gaussian function.

**Gaussian Pulse**

Consider an excitation of Gaussian pulse with finite duration, i.e.

\[ F_w(\tilde{C}) = e^{-c|\tilde{t}|^2}. \]  

**Monochromatic Wave**

In case of a monochromatic wave at the FEL resonant frequency:

\[ F_w(\tilde{C}) = \delta(\tilde{C}). \]  

and eq. (15), (17), (18) and (31) give:


DISCUSSION

Because the amplification of the plane wave in an FEL with infinitely wide beam depends on its propagation angle via changing the detuning from the FEL resonance (9): \( \tilde{C}_{\text{ad}} \equiv \tilde{C} - \tilde{k}_0^2 \). One can expect that this will confine effective amplification to a narrow cone alone that axis of the FEL and, therefore, some optical guiding of the optical beam. Our numerical studies and analytical estimates showed that in a typical FEL this effect results in near-linear RMS size grows. Even though the growth of the transverse beam size is smaller than in free space case, the optical guiding FEL effect by an infinite electron beam is much smaller and is different from that by a beam with finite size.

SUMMARY

We obtained results resembling some aspects of the 1D theory, especially for the longitudinal dynamics. However, we successfully incorporated diffraction into the evolution of the transverse density modulation profile, which is of critical importance for studying the transverse coherence of the electron beam in CeC. For few selected initial conditions, the spatial domain solutions were expressed through 1D or 2D integrals, which can be readily numerically integrated.

We showed that while FEL dispersion provides some optical guiding, it is very different from that provided by the finite size electron beams.

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REFERENCES